Communications in Mathematical Research
28(2)(2012), 121–126

Stability of Fredholm Integral Equation of the First Kind in Reproducing Kernel Space*

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Communicated by Ma Fu-ming

Abstract: It is well known that the problem on the stability of the solutions for Fredholm integral equation of the first kind is an ill-posed problem in $C[a,b]$ or $L^2[a,b]$. In this paper, the representation of the solution for Fredholm integral equation of the first kind is given if it has a unique solution. The stability of the solution is proved in the reproducing kernel space, namely, the measurement errors of the experimental data cannot result in unbounded errors of the true solution. The computation of approximate solution is also stable with respect to $\|\cdot\|_C$ or $\|\cdot\|_{L^2}$. A numerical experiment shows that the method given in this paper is stable in the reproducing kernel space.

Key words: Freholm integral equation, ill-posed problem, reproducing kernel space
2000 MR subject classification: 74S30
Document code: A
Article ID: 1674-5647(2012)02-0121-06

1 Introduction

The Fredholm integral equation of the first kind is of the form

$$\int_a^b k(x,y)u(y)\mathrm{d}y = f(x), \qquad a \le x \le b. \tag{1.1}$$

It is well known that the problem on the stability for Fredholm integral equation of the first kind is an ill-posed problem in $C[a,b]$ or $L^2[a,b]$. Some related works can be found in [1–6]. Namely, when given the right-hand side $f(x)$ a perturbation, it could be caused large errors of solution $u(y)$ in $L^2[0,\pi]$.

Many problems in science and engineering lead to seeking for the solution of the first kind of linear integral equations. In [1, 7], the 1D heat conduction equation with initial and

*Received date: Apirl 26, 2006.
Foundation item: NSF (A201015) of Heilongjiang Province.

boundary conditions

$$\begin{cases} v_t(x,t) = v_{xx}(x,t), & 0 < x < \pi,\ t > 0, \\ v(0,t) = v(\pi,t) = 0, & t > 0, \\ v(x,0) = u(x), & 0 \le x \le \pi \end{cases} \tag{1.2}$$

is given. The solution of (1.2) is

$$v(x,t) = \sum_{n=1}^{\infty} a_n \mathrm{e}^{-n^2 t} \sin(nx),$$

where $a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} u(y)\sin(ny)\mathrm{d}y$.

The inverse heat conduction problem involves determining $u(x)$ from the given data $v(\,\cdot\,,\,t)$. Then the problem is to solve the solution of the Fredholm integral equation of the first kind

$$\int_0^{\pi} k(x,y)u(y)\mathrm{d}y = v(x,t), \qquad 0 \le x \le \pi, \tag{1.3}$$

where

$$k(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \mathrm{e}^{-n^2 t}\sin(nx)\sin(ny).$$

In this paper, the representation of the solution is obtained for Fredholm integral equation of the first kind in the reproducing kernel space $W_2^1[a,b]$. The reproducing kernel space $W_2^1[a,b]$ was defined in [8]. The computation of approximate solution is also stable when a perturbation is convergent to zero in the sense of $\|\cdot\|_C$ or $\|\cdot\|_{L^2}$ in the reproducing kernel space. We illustrate a numerical experiment in the last section of this paper.

2 The Solution of (1.1)

In this section, if the solution of (1.1) is unique, then the representation of the solution is given in the reproducing kernel space for the Fredholm integral equation of the first kind as follows:

$$(Au)(x) \triangleq \int_a^b K(x,s)u(s)\mathrm{d}s = f(x), \qquad u, f \in W_2^1[a,b], \tag{2.1}$$

where $W_2^1[a,b]$ is defined in [8], $u(x)$ is a determined function, $f(x)$ is a given function and the kernel $K(x,s)$ satisfies the conditions

$$\iint_{[a,b]\times[a,b]} |K(x,s)|^2 \mathrm{d}x\mathrm{d}s \le M_1, \qquad M_1 \in \mathbf{R}, \tag{2.2}$$

and

$$\iint_{[a,b]\times[a,b]} \Big|\frac{\partial K(x,s)}{\partial x}\Big|^2 \mathrm{d}x\mathrm{d}s \le M_2, \qquad M_2 \in \mathbf{R}. \tag{2.3}$$

Lemma 2.1 *The operator A defined in* (2.1) *is a bounded linear operator from $W_2^1[a,b]$ to $W_2^1[a,b]$ under the conditions* (2.2) *and* (2.3).

In order to obtain the representation of the solution of (2.1), set the reproducing kernel $R_y(x)$ in $W_2^1[a,b]$ as

$$R_y(x) = \frac{1}{2\sinh(b-a)}[\cosh(x+y-b-a) + \cosh(|x-y|-b+a)] \tag{2.4}$$