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ERROR ESTIMATES FOR TWO-SCALE COMPOSITE FINITE ELEMENT APPROXIMATIONS OF NONLINEAR PARABOLIC EQUATIONS*

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Abstract

We study spatially semidiscrete and fully discrete two-scale composite finite element method for approximations of the nonlinear parabolic equations with homogeneous Dirichlet boundary conditions in a convex polygonal domain in the plane. This new class of finite elements, which is called composite finite elements, was first introduced by Hackbusch and Sauter [Numer. Math., 75 (1997), pp. 447-472] for the approximation of partial differential equations on domains with complicated geometry. The aim of this paper is to introduce an efficient numerical method which gives a lower dimensional approach for solving partial differential equations by domain discretization method. The composite finite element method introduces two-scale grid for discretization of the domain, the coarse-scale and the fine-scale grid with the degrees of freedom lies on the coarse-scale grid only. While the fine-scale grid is used to resolve the Dirichlet boundary condition, the dimension of the finite element space depends only on the coarse-scale grid. As a consequence, the resulting linear system will have a fewer number of unknowns. A continuous, piecewise linear composite finite element space is employed for the space discretization whereas the time discretization is based on both the backward Euler and the Crank-Nicolson methods. We have derived the error estimates in the $L^\infty(L^2)$-norm for both semidiscrete and fully discrete schemes. Moreover, numerical simulations show that the proposed method is an efficient method to provide a good approximate solution.

Mathematics subject classification: 35J20, 65N15, 65N30.
Key words: Composite finite elements, Nonlinear parabolic problems, Coarse-scale, Fine-scale, Semidiscrete, Fully discrete, Error estimate.

1. Introduction

The main purpose of this work is to formulate and study the semidiscrete and fully discrete composite finite element (CFE) approximations of solutions of nonlinear parabolic problem with the Dirichlet boundary condition. In [21], Rech, Sauter and Smolianski have studied CFE method for elliptic problems with Dirichlet boundary conditions, where they have improved upon the error estimation using finer-scale discretization for the slave nodes. We consider the following model nonlinear parabolic initial-boundary value problem, for $u = u(x,t)$,

$$\begin{aligned} u_t - \nabla\cdot(a(u)\nabla u) &= f(u) \quad \text{in } \Omega\times J,\ J=(0,T], \\ u &= 0 \quad \text{on } \Gamma\times J, \\ \text{with } u(\cdot,0) &= v \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

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where, Ω is a bounded domain in $\mathbb{R}^2$ with Lipschitz boundary Γ, $v : \mathbb{R}^2 \to \mathbb{R}$, $a : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are given functions. Let the known smooth functions $a(u)$ and $f(u)$ satisfy the following condition:

$$0 < \mu \leq a(u) \leq M, \quad |a'(u)| + |f'(u)| \leq B \quad \text{for } u \in \mathbb{R}. \tag{1.2}$$

Under the above conditions, we assume that the above problem admits a unique solution which is sufficiently smooth.

The CFE method was initially introduced for coarse level discretizations of boundary value problems (cf. [9–11]). The purpose of this paper is to generalize certain known error estimates of two-scale CFE approximation for elliptic problems to the time dependent parabolic initial and boundary value problems. In standard finite element method (FEM), the usual requirement is that the underlying finite element mesh has to resolve the domain boundary. However, discretization of the domain with a coarse-scale mesh includes huge number of unknowns. In other words, it increases the dimension of the finite element space. Therefore, we introduces the CFE method via two-scale grid discretization: In the interior of the domain at a proper distance from the boundary the solution is approximated by the coarse-scale parameter H and the vicinity of the boundary is discretized by the fine-scale parameter h to approximate the Dirichlet boundary conditions until a sufficient approximation quality is reached [8, 24]. The CFE method can be illustrated as a generalization of standard finite elements, where we approximate the Dirichlet boundary conditions in a flexible adaptive manner [6, 14]. The degrees of freedom (Dofs) lies only on the coarse-scale grid in the interior of the domain. This causes a reduction of dimension of the CFE space, which is very advantageous in contrast to the standard FEM.

In this paper, we consider two-scale CFE method for the domain discretization of solving nonlinear parabolic problems (1.1), which allows low-dimensional discretization for the domain while the convergence rates are preserved. In the finite element literature the term "composite" has also appeared in *composite triangles* (cf. [7, 27]). Our approach is to introduce composite finite elements for an adaptive approximation of Dirichlet boundary conditions. Composite finite elements build the necessary adaptation into basis functions. Far from the domain boundary, the basis functions coincide with the standard basis functions on the structured grid. In the vicinity of the boundary, the standard basis is modified to resolve the shape of the domain boundary. Related approaches for coarsening the meshes can be found in [3, 13, 29]. The CFE method for parabolic problems in both convex and nonconvex domains has been extensively studied, see [16, 18, 19]. The convergence properties in the $L^\infty(L^2)$-norm for both semidiscrete and fully discrete methods are derived and analyzed. To the best of author's knowledge, the two-scale CFE method for solving nonlinear parabolic problems in convex domains is being reported for the first time in the literature.

Outline of the article. The paper is organized as follows. Section 2 introduces the CFE discretization and locating the degrees of freedom as well as the slave nodes. The extrapolation operator is defined in this section in order to construct the CFE space. Section 3 is devoted to construct the CFE space. In this section, we construct the CFE nodal basis function and show the existence and uniqueness of the CFE solution for the spatially semidiscrete approximation of the given problem (1.1). In section 4, we derive the error estimates for the spatially semidiscrete scheme. Section 5 provides the fully discrete error analysis for both the backward Euler and the Crank-Nicolson schemes. Also, the error analysis for linearized modification in both the backward Euler and the Crank-Nicolson schemes are presented in this section. Section 6 presents