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Fixing the Residual Flattening of an Upwind Compact Scheme for Steady Incompressible Flows in Enclosed Domains

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Abstract. The iterative convergence of the upwind compact finite difference scheme for the artificial compressibility method [A. Shah *et al.*, A third-order upwind compact scheme on curvilinear meshes for the incompressible Navier-Stokes equations, Commun. Comput. Phys. **5** (2009)] is studied. It turns out that for steady flows in enclosed domains the residuals do not converge to machine zero. The reason is a non-uniqueness of the calculated pressure in the case where Neumann boundary conditions for the pressure are imposed on all boundaries. The problem can be fixed by modifying the derivatives of mass flux obtained from the upwind compact scheme to satisfy the global mass conservation constraint. Numerical tests show that with this modification the scheme converges to machine zero with the original third-order accuracy.

AMS subject classifications: 76D05, 65M06

Key words: Incompressible Navier-Stokes equations, artificial compressibility, upwind compact difference, convergence, enclosed domain.

1. Introduction

The artificial compressibility (AC) method was proposed by Chorin [6] for the numerical solution of the incompressible Navier-Stokes equations. In this method, a pseudo-time derivative of pressure is added to the continuity equation, so that the original elliptic-parabolic system of equations become hyperbolic in time. After that, various well-established compressible flow numerical algorithms can be used in the AC method. The AC method was initially used to compute steady flows with the approximate factorisation algorithms [20, 32]. Later it was extended to unsteady incompressible flows [2, 34, 41] by using the

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dual time stepping technique [5, 16]. Meanwhile, high-resolution total variation diminishing (TVD) schemes [15] and high-order flux difference splitting (FDS)-based upwind schemes [33–35] were introduced in conjunction with the lower-upper symmetric Gauss-Seidel (LU-SGS) scheme [44], line Gauss-Seidel relaxation scheme [33,34] and generalised minimal residual (GMRES) algorithm [13, 31]. The influence of the artificial pressure wave of the hyperbolic system on convergence of the AC method was analysed by Kwak *et al.* [20, 21]. Recently, high-order discontinuous Galerkin schemes have been also incorporated in this method [48].

Compact finite difference schemes attracted substantial attention since they have lower truncation errors and higher spectral resolution than non-compact ones [22]. For compressible flows such methods are often employed in combination with weighted essentially non-oscillatory (WENO) schemes [18] in order to deal with shock waves. Thus Deng [7] developed a WENO reconstruction-based compact nonlinear scheme, Jiang *et al.* [17] considered a WENO-weighted compact difference scheme, and Pirozzoli [27] and Ren *et al.* [28] worked with conservative compact reconstruction-WENO hybrid schemes, to mention a few. These methods have better accuracy and resolution than stand-alone WENO schemes. For incompressible flows, the solutions of the corresponding equations have no strong discontinuities, so that any linearly stable compact scheme can be exploited. In particular, central compact schemes with implicit central filtering [43] are applied to the AC method [8, 29, 47]. The FDS scheme [30] is a full wave approximate Riemann solver for the compressible Euler equations and can capture shear waves accurately. Following the successful application of FDS-based high-order upwind schemes in incompressible flow simulations [33–35], Shah *et al.* [37, 38] developed FDS-based third- and fifth-order upwind compact schemes, and demonstrated their superior spectral resolution over the FDS-based non-compact upwind schemes of same order [39].

However, the authors of this work discovered that although for steady flows in enclosed domain the residuals of the conservative non-compact upwind schemes of [33–35] converge to machine zero, the nonconservative third-order upwind compact schemes of [37, 38] converge to a number much greater than machine zero. In monitoring the convergence history of the flow variables of the scheme [37] for steady flows in enclosed domain, we noted that the velocity increments converge to machine zero but the pressure increments converge to a number much greater than machine zero. The reason is the non-uniqueness of the pressure in enclosed domains, and we emphasize that this has an adverse impact on the convergence of the pressure time derivative term in the continuity equation of the AC method. In order to fix the problem, we use modified numerical derivatives of the mass flux to satisfy the global mass conservation constraint. Numerical examples show that the residuals of the modified upwind compact scheme for steady flows in enclosed domain converge to machine zero with the third-order accuracy.

The rest of the paper is organised as follows. Section 2 reviews the governing equations and FDS-based third-order upwind compact scheme. The convergence of this scheme for steady flows is tested and analysed in Section 3, and a modification of the residual stall is suggested in Section 4. Benchmark examples in Section 5 show the effectiveness of the modified scheme. Finally, our conclusion is given in Section 6.