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ANALYSIS ON A NUMERICAL SCHEME WITH SECOND-ORDER TIME ACCURACY FOR NONLINEAR DIFFUSION EQUATIONS*

Xia Cui
Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009-26, Beijing 100088, China
Email: cui_xia@iapcm.ac.cn
Guangwei Yuan[1)]
Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, P.O. Box 8009-26, Beijing 100088, China
Email: yuan_guangwei@iapcm.ac.cn
Fei Zhao
College of Science, North China University of Technology, Beijing 100144, China
Graduate School of China Academy of Engineering Physics, Beijing 100088, China
Email: zhaofei17@gscaep.ac.cn

Abstract

A nonlinear fully implicit finite difference scheme with second-order time evolution for nonlinear diffusion problem is studied. The scheme is constructed with two-layer coupled discretization (TLCD) at each time step. It does not stir numerical oscillation, while permits large time step length, and produces more accurate numerical solutions than the other two well-known second-order time evolution nonlinear schemes, the Crank-Nicolson (CN) scheme and the backward difference formula second-order (BDF2) scheme. By developing a new reasoning technique, we overcome the difficulties caused by the coupled nonlinear discrete diffusion operators at different time layers, and prove rigorously the TLCD scheme is uniquely solvable, unconditionally stable, and has second-order convergence in both space and time. Numerical tests verify the theoretical results, and illustrate its superiority over the CN and BDF2 schemes.

Mathematics subject classification: 65M06, 65M12, 65M15.
Key words: Nonlinear diffusion problem, Nonlinear two-layer coupled discrete scheme, Second-order time accuracy, Property analysis, Unique existence, Convergence.

1. Introduction

Nonlinear diffusion problems appear in radiation hydrodynamic, reservoir geomechanics, astrophysics, and many other scientific fields. Efficient and accurate numerical simulations for them play an essential role in the associated research areas [13, 26]. To solve nonlinear diffusion equations, people usually use time implicit discretizations, since they do not suffer severe restrictions on the time step length while such restrictions are required by operator splitting methods and time explicit methods due to numerical stability. For example, in [8,21,23] Crank-Nicolson (CN) schemes are studied and predictor-corrector and extrapolation techniques

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1) Corresponding author.

are employed to linearize the numerical procedure. In [2, 14, 16, 22] the backward Euler (BE) and linear CN schemes are studied for some nonlinear diffusion systems in various applications. Compared with linear semi-implicit schemes, nonlinear fully implicit (FI) schemes are much more appealing since they can capture the transient status more precisely and avoid some nonphysical phenomena in case physical quantities vary violently with time advance [12,25–27].

Compared with the first-order time evolution approaches such as the BE method, the second-order ones are highly desirable to obtain efficient and accurate solutions (see, e.g., [18–20]). To solve multi-dimensional heat conduction and radiation heat transfer problems, some numerical schemes with second-order accuracy in both space and time are developed, e.g., in [6, 7] and [19]. Specifically, in [19] some efficient numerical methods with second-order time discretization are studied for non-equilibrium radiation diffusion problem. It shows that although being unconditionally stable in discrete L^2 sense and second-order time accurate, the CN scheme causes nonphysical numerical oscillations when using large time step length; while the second-order BE time discretization (in literatures it is sometimes referred to as BDF2, backward difference formula second-order) does not present such oscillations, hence permits larger time step length, and has significantly higher accuracy and efficiency than the first-order BE time discretization. In [5–7], a discrete scheme with two-layer coupled discretization (TLCD) at each time step is presented, wherein in addition to primary unknowns at integer time layer, new primary unknowns at half time layer are introduced. Hence the TLCD scheme is a two-layer coupled discrete system at each time step. Numerical tests show that it can avoid the apparent numerical oscillation, which differs from CN scheme.

To carry out theoretical studies on numerical methods for linear and nonlinear PDEs, some discrete functional analysis tools have been developed in [1, 9, 11] and [31, 32]. They are very useful in establishing fundamental properties of some discrete schemes and iterative methods for diffusion equations in divergence or non-divergence form, e.g., [3,10,24,25,32]. Some first-order BE schemes are theoretically analyzed in [2, 16, 22], etc. The CN and BDF2 schemes are analyzed respectively in [8,14,21,23] and [3,4,28], etc. Especially, by developing a temporal-spatial error splitting argument technique proposed in [15,16], the authors of [14,17,21,30] successfully analyze the properties of some semi-implicit (linear) schemes for nonlinear parabolic problems without mesh ratio requirement. However theoretical analysis on fully implicit (nonlinear) schemes for nonlinear diffusion problems is still rather rare. In the analysis on nonlinear discrete schemes for diffusion operators in divergence form, the main difficulty lies in the estimate of the nonlinear diffusion term.

This paper contributes to presenting theoretical analysis on the numerical performance for the TLCD scheme. No such analysis result has been published to our knowledge. A new difficulty occurs, which is caused by the additional terms due to the coupling of two-layer discrete equations. They contain discrete diffusion operators whose coefficients are identical to the difference between the values of the diffusion coefficient at different time layers. To focus on illustrating the main ideas on the theoretical analysis, a finite difference method is considered for spatial discretization. Actually, other discrete ways such as finite element and finite volume methods are also applicable. First, we prove the existence of the solution for the TLCD scheme with a fixed point theorem. A novelty in this procedure is that we tactfully bound the TLCD solution and the difference between the solutions at two layers in proper discrete norms simultaneously. Next, with this boundedness result, a new reasoning technique is developed to overcome the difficulty arising from the additional terms, and a rigorous proof for the convergence of the TLCD scheme is performed. Then, the uniqueness of its solution is