INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 5, Pages 656–673



# SUPERCONVERGENCE AND FLUX RECOVERY FOR AN ENRICHED FINITE ELEMENT METHOD

CHAMPIKE ATTANAYAKE AND SO-HSIANG CHOU

**Abstract.** We introduce a flux recovery scheme for an enriched finite element method applied to an interface diffusion equation with absorption. The method is a variant of the finite element method introduced by Wang *et al.* in [20]. The recovery is done at nodes first and then extended to the whole domain by interpolation. In the case of piecewise constant diffusion coefficient, we show that the nodes of the finite elements are superconvergence points for both the primary variable $p$ and its flux $u$. In particular, in the absence of the absorption term zero error is achieved at the nodes and interface point in the approximation of $u$ and $p$. In the general case, pressure error at the nodes and interface point is second order. Numerical results are provided to confirm the theory.

**Key words.** Flux recovery technique, superconvergence, enriched finite element, immersed finite element method.

## 1. Introduction

We consider the interface two-point boundary value problem

$$\begin{cases} -(\beta(x)p'(x))' + w(x)p(x) = f(x), & x \in I = (a,b), \\ p(a) = p(b) = 0, \end{cases} \tag{1}$$

where $w(x) \geq 0$, and $0 < \beta \in C[a,\alpha] \cup C[\alpha,b]$ is discontinuous across the interface $\alpha$ with the jump conditions

$$[p]_\alpha = 0, \tag{2}$$

$$[\beta p']_\alpha = g. \tag{3}$$

Here the unknown function $p$ may stand for the pressure or temperature in a medium with certain physical properties and the derived quantity $u := -\beta p'$ is the corresponding Darcy velocity or heat flux, which is equally important. The piecewise continuous $\beta$ reflects a nonuniform material or medium property and the function $w(x)$ reflects the surroundings of the material. Problem (1) can also be viewed as the steady neutron diffusion problem [19]. However, in this paper we will refer to $p$ as pressure. Due to its one-dimensional simple structure, many mathematical and numerical properties of related numerical methods can be explicitly worked out. For example, in this paper the flux jump $g$ in (3) will be taken as zero. In fact, if $g \neq 0$ we can handle the nonhomogenous flux jump condition as follows[1]. Let $\tilde{p}$ be a boundary vanishing function such that $[\tilde{p}]_\alpha = 0$ and $[\beta\tilde{p}']_\alpha = g$. Among all possible $\tilde{p}$, we can choose those suitable for our numerical computation as well. For instance,

$$\tilde{p}(x) = \begin{cases} 0, & a \leq x < \alpha, \\ -\frac{g}{\beta^+(b-\alpha)}(x-\alpha)^2 + \frac{g}{\beta^+}(x-\alpha), & \alpha \leq x \leq b, \end{cases} \tag{4}$$

Received by the editors October 23, 2020 and in revised form, June 22, 2021.
2000 *Mathematics Subject Classification.* 65N15, 65N30.

where $\beta^+ = \lim_{x\to\alpha^+} \beta(x)$. A transformed new variable $p - \tilde{p}$ will then give rise to an interface problem with homogeneous jump conditions. One can generalize the above technique to higher dimensions with the help of those used in the immersed finite element shape functions construction. In general, it is very instructive to study problem (1) before moving to its higher dimensional and/or nonsteady state versions. It is in this spirit that we shall study the associated enriched finite element approximation. Recent studies of immersed finite element and volume methods on similar one-dimensional problems can be found in [4, 5].

Numerical methods for the interface problem (1) generally use meshes that are either fitted or unfitted with the interface. A method allowing unfitted meshes would be very efficient when one has to follow a moving interface in a temporal problem. For an in-depth exposition of the numerics and applications of interface problems, we refer the readers to [14] and the references therein. For our purpose here let us only mention two classes of methods: (a) the class of immersed finite element and difference methods and (b) the class of enriched finite element methods. For example, in an immersed finite element (IFE) method, the mesh is made up of interface elements where the interface intersects elements (thus immersed) and noninterface elements where the interface is absent. On a noninterface element one uses standard local shape functions, whereas on an interface element one uses piecewise standard local shape functions subject to continuity and jump conditions. Representative works on IFE methods can be found in [12, 13, 14, 15, 16, 18], among others.Recent advances in the subject of superconvergence of the IFE method are [9, 10] and the related references there in. For the enriched method, the standard finite element method is enriched with some nonstandard elements that reflect the presence of the interface. It was originally designed to handle crack problems [2, 8, 17], but for recent years efforts have been made to generalize it to fluid problems, see [20] and the references therein.

In this paper, we are interested in studying a flux recovery procedure for an enriched finite element. The procedure can produce accurate approximate flux $u_h$ of $p$, once an approximate $p_h$ has been obtained. It is important that the procedure can recover flux without having to solve any system of equations. Chou and Tang [7] initiated such methods when the mesh is fitted. Later it was generalized to the immersed interface mesh case using linear immersed finite elements (IFE) of Li *et al.* [16] and their variants for one dimensional elliptic and parabolic problems [1, 6]. In this paper we extend the methodology to enriched finite elements from the conforming $P_1$ elements.

The idea of the flux recovery scheme in [7] is very easy to describe in the one dimensional case. Suppose let there be given an expression of the exact flux $u(x_i)$ at some mesh point $x_i$ in terms of a weighted integral of $p$, which can be obtained as follows. Let $\phi$ be a function with compact support $K$ such that $I_i = [x_{i-1}, x_i] \subset K$, the interface point $\alpha \notin K$ (non-interface element), $\phi(x_{i-1}) = 0$, $\phi(x_i) = 1$. An example of such a function is the standard finite element hat function. Multiplying (1) by $\phi$ and integrating by parts, we see that the flux $u$ satisfies

$$u(x_i) = -\int_{I_i} \beta p'\phi' dx - \int_{I_i} wp\phi dx + \int_{I_i} f\phi dx.$$

It is then natural to define an approximate flux $u_h$ at $x_i$ as

$$u_h(x_i) = -\int_{I_i} \beta p_h'\phi' dx - \int_{I_i} wp_h\phi dx + \int_{I_i} f\phi dx.$$