INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 5, Pages 674–689

AN EFFICIENT NONLINEAR SOLVER FOR STEADY MHD BASED ON ALGEBRAIC SPLITTING

MENGYING XIAO

Abstract. We propose a new, efficient, nonlinear iteration for solving the steady incompressible MHD equations. The method consists of a careful combination of an incremental Picard iteration, Yosida splitting, and a grad-div stabilized finite element discretization. At each iteration, the Schur complement remains the same, is SPD, and can be easily and effectively preconditioned with the pressure mass matrix. Furthermore, this method decouples the block Schur complement into 2 simple Stokes Schur complement. We show that the iteration converges linearly to the discrete MHD system solution, both analytically and numerically. Several numerical tests are given which reveal very good convergence properties, and excellent results on a benchmark problem.

Key words. Steady MHD, algebraic splitting, incremental Picard Yosida method, nonlinear solver.

1. Introduction

Magnetohydrodynamics (MHD) describes the flow of electronically conducting fluids in a magnetic field, which arises in a wide range of applications such as geophysics and astrophysics [2, 3, 5, 6, 10]. We herein develop an efficient nonlinear iteration scheme to solve steady MHD in a convex domain Ω, which is given by

$$-\nu\Delta u + (u\cdot\nabla)u - s(B\cdot\nabla)B + \nabla\bar{p} = f, \tag{1}$$

$$\nabla\cdot u = 0, \tag{2}$$

$$-\nu_m\Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u - \nabla\lambda = \nabla\times g, \tag{3}$$

$$\nabla\cdot B = 0, \tag{4}$$

where u is the velocity of fluid, $\bar{p}$ is a modified pressure, B is the magnetic field, λ is a variable acting as a Lagrange multiplier corresponding to the solenoidal constraint on the magnetic field, f is the body forcing, $\nabla\times g$ is the forcing on the magnetic field B, s is a coupling number, ν is the kinematic viscosity and ν_m is the magnetic diffusivity. For simplicity we consider homogeneous Dirichlet boundary conditions for both u and B and $s = 1$. With minor changes, our analysis will also hold for inhomogeneous or periodic boundaries, as well as no slip velocity together with $B\cdot n = 0$ and $(\nabla\times B)\times n = 0$ (in this case the Maxwell equation uses the curl curl form of the dissipation term).

Although MHD couples the Navier-Stokes equations (NSE) for fluids to Maxwell's equations for electromagnetics, the linear systems that arise have similar structure to those of the NSE, but in block form. Using Picard's method to solve steady MHD equations requires solving a linear saddle point system in each iteration. Difficulties arise when solving such saddle point systems, such as how to build an efficient preconditioner for iterative linear solvers for large problems [4], and how to derive a robust error estimator [11]. Several approximations are made to solving this saddle point linear system. The linear algebra problem is actually worse in the steady case, than the time dependent case, since one cannot take advantage of

Received by the editors March 1, 2020.
2020 *Mathematics Subject Classification.* 37N30, 37M05, 65Y04.

traditional splitting methods such as projection methods, or lag nonlinear terms in a temporal discretization.

We herein propose a method to solve the nonlinear system (1)-(4) based on an algebraic splitting method shown to work very well for a NSE system in [13], which will require much easier linear system solvers than standard nonlinear solvers do. It is based on a careful combination of an incremental Picard iteration, grad-div stabilization, and algebraic splitting of Yosida-Type. A derivation of the method is given below.

The standard Picard iteration scheme for (1)-(4) is given below: Guess u_0, B_0 and for $k = 1, 2, \dots$, and find u_k, p_k, B_k, λ_k satisfying

$$-\gamma\nabla(\nabla\cdot u_k) + u_{k-1}\cdot\nabla u_k - B_{k-1}\cdot\nabla B_k + \nabla p_k - \nu\Delta u_k = f, \tag{5}$$

$$\nabla\cdot u_k = 0, \tag{6}$$

$$-\gamma_m\nabla(\nabla\cdot B_k) + u_{k-1}\cdot\nabla B_k - B_{k-1}\cdot\nabla u_k - \nabla\lambda_k - \nu_m\Delta B_k = \nabla\times g, \tag{7}$$

$$\nabla\cdot B_k = 0. \tag{8}$$

Although $\nabla(\nabla\cdot u_k) = \nabla(\nabla\cdot B_k) = 0$ due to (6) and (8), when discretized with common finite element choices, such as Taylor-Hood, we only have weak enforcement of (6) and (8). Thus parameters γ, γ_m penalize the divergence error of numerical solutions. Notice these grad-div stabilization terms can make problem unstable if gammas are too large as they are singular. In practice, $\gamma, \gamma_m \sim 1$ are close to optimal parameters. Adding increments $-\nabla p_{k-1}$ and $\nabla\lambda_{k-1}$ on both sides of (5) and (7) respectively, gives an incremental Picard iteration:

$$-\gamma\nabla(\nabla\cdot u_k) + u_{k-1}\cdot\nabla u_k - B_{k-1}\cdot\nabla B_k + \nabla\delta_k^p - \nu\Delta u_k = f - \nabla p_{k-1}, \tag{9}$$

$$\nabla\cdot u_k = 0, \tag{10}$$

$$-\gamma_m\nabla(\nabla\cdot B_k) + u_{k-1}\cdot\nabla B_k - B_{k-1}\cdot\nabla u_k - \nabla\delta_k^\lambda - \nu_m\Delta B_k = \nabla\times g + \nabla\lambda_{k-1}, \tag{11}$$

$$\nabla\cdot B_k = 0. \tag{12}$$

which is equivalent to the usual Picard iteration (assuming $p_0 = \lambda_0 = 0$). After applying a finite element discretization to (9)-(12), a block linear system arises at each iteration in the form

$$\begin{pmatrix} A_1 & N_1 & C_1^T & 0 \\ N_1 & A_2 & 0 & C_1^T \\ C_1 & 0 & 0 & 0 \\ 0 & C_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{B} \\ \bar{\delta^p} \\ \bar{\delta^\lambda} \end{pmatrix} = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

where A_1 and A_2 consist of a stiffness matrix K, divergence matrix E and contributions of convection terms, N_1 is the contribution of convection term from Maxwell's equation, and C_1 is a rectangular matrix coming from (10) or (12). The *bar* notation denotes the coefficient vectors corresponding to the associated finite element functions. This block linear system, just like for the NSE, takes the form of a saddle point system

$$\begin{pmatrix} A & C^T \\ C & 0 \end{pmatrix} \begin{pmatrix} \bar{X} \\ \bar{Y} \end{pmatrix} = \begin{pmatrix} \tilde{F} \\ 0 \end{pmatrix}, \tag{14}$$

if we set $A = \begin{pmatrix} A_1 & N_1 \\ N_1 & A_2 \end{pmatrix}$, $C = \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix}$, $\bar{X} = \begin{pmatrix} \bar{u} \\ \bar{B} \end{pmatrix}$, $\bar{Y} = \begin{pmatrix} \bar{\delta^p} \\ \bar{\delta^\lambda} \end{pmatrix}$, $\tilde{F} = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix}$. Such a system is well known to be very difficult to solve. Direct solvers are not