INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 6, Pages 758–787

COLLOCATION METHODS FOR A CLASS OF INTEGRO-DIFFERENTIAL ALGEBRAIC EQUATIONS

HAIYAN ZHANG AND HUI LIANG*

Abstract. A class of index-1 integro-differential algebraic equations modeling a hydraulic circuit that feed a combustion process is considered. The existence, uniqueness and regularity are analyzed in detail. Two kinds of collocation methods are employed to solve the equation numerically. For the first one, the derivative and algebraic components are approximated in globally continuous and discontinuous polynomial spaces, respectively; and for another one, both the derivative and algebraic components are solved in globally continuous piecewise polynomial spaces. The convergence, global and local superconvergence are described for these two classes of collocation methods. Some numerical experiments are given to illustrate the obtained theoretical results.

Key words. Integro-differential algebraic equations, tractability index, regularity, collocation methods, convergence analysis.

1. Introduction

Integro-differential algebraic equations (IDAEs) arise in many mathematical modeling processes, for example, [6, 7] (Kirchhoff's laws); [10] (circuit simulation); [9] (the seat-occupant dynamic model); and [16] (hydraulic circuit that feeds a combustion process). Due to the rich applications, there are many researchers focus on this research area. In [12], the convergence properties of implicit Runge-Kutta methods of Pouzet-type for IDAEs that arise when solving singularly perturbed Volterra integro-differential equations are analyzed; in [1], the global and local superconvergence properties of piecewise polynomial collocation solutions for index-1 semi-explicit IDAEs are discussed; various aspects of the numerical treatment of IDAEs are studied in [2, 3, 4, 5] (existence and uniqueness of analytic solutions of certain IDAEs; convergence of the implicit Euler method and methods based on backward differentiation formulas (BDFs)); [18] (well-posedness results for non-autonomous integro-differential-algebraic evolutionary problems); and [17] (convergence of the Legendre spectral Tau-method); in [15], the tractability index of IDAEs are defined, the given IDAEs system of index 1 is decoupled into the inherent system of regular Volterra integro-differential equations and a system of second-kind Volterra integral equations, and the convergence, global and local superconvergence are studied for two kinds of collocation methods.

Motivated by [16], in this paper, we consider the following IDAE comes from a hydraulic circuit that feed a combustion process:

(1)
$$\begin{cases} y'(t) + b_{11}(t)y(t) + b_{12}(t)z(t) = f(t) + \displaystyle\int_0^t \left[K_{11}(t,s)y(s) + K_{12}(t,s)z(s)\right] ds, \\ \qquad\quad b_{21}(t)y(t) + b_{22}(t)z(t) = g(t), \end{cases}$$

where $t \in I := [0,T]$, the given functions $b_{pq}, K_{1q} \in \mathbb{R},\ p,q = 1,2$, and $|b_{22}(t)| \geq b_0 > 0$. The system (1) is complemented by a given set of initial values $(y(0), z(0))^T =$

Received by the editors May 22, 2020 and, in revised form, September 3, 2021.
2000 *Mathematics Subject Classification.* 45J99, 65R99.
*Corresponding author.

$(y_0, z_0)^T$, which is assumed to be consistent, i.e.,

$$b_{21}(0)y(0) + b_{22}(0)z(0) = g(0).$$

This paper is aimed at the IDAE (1), and the outline is as follows. In Section 2, we recall the tractability index of IDAEs, and check the index for the IDAE (1). In Section 3, we analyze the existence, uniqueness and regularity of the analytic solution. The collocation scheme, convergence and global (local) superconvergence results in different collocation spaces are shown in Section 4, and the corresponding results in the same collocation space are shown in Section 5. In Section 6, we give some numerical examples to illustrate the theoretical results obtained in this paper.

2. The tractability index of IDAEs

In this section, we will check that (1) is index-1 tractable. For this purpose, we first review the definition of the tractability index induced in [15] for the following general linear IDAE:

$$A(t)x'(t) + B(t)x(t) + \int_0^t K(t,s)x(s)\,ds = F(t), \tag{2}$$

where $A, B, K \in \mathbb{R}^{d\times d}$ and $F \in \mathbb{R}^d$. Before stating the definition of the tractability index for IDAEs (2), we first recall the definition of the notion of ν-smoothing of the linear Volterra integral operator $\mathscr{V} : C(I) \to C(I)$ defined by

$$(\mathscr{V}x)(t) := \int_0^t K(t,s)x(s)\,ds,\ t \in I, \tag{3}$$

with the continuous matrix kernel

$$K(t,s) := [K_{pq}(t,s)] \in \mathbb{R}^{d\times d}.$$

Definition 1. *(see [13]) The Volterra integral operator $\mathscr{V}$ in (3) is said to be ν-smoothing if there exist integers $\nu_{pq} \geq 1$ with*

$$\nu := \max_{1\leq p,q\leq d}\{\nu_{pq}\},$$

such that

(a): $\left.\dfrac{\partial^j K_{pq}(t,s)}{\partial t^j}\right|_{s=t} = 0,\ t \in I,\ j = 0, 1, \ldots, \nu_{pq} - 2;$

(b): $\left.\dfrac{\partial^{\nu_{pq}-1} K_{pq}(t,s)}{\partial t^{\nu_{pq}-1}}\right|_{s=t} \neq 0,\ t \in I;$

(c): $\dfrac{\partial^{\nu_{pq}} K_{pq}(t,s)}{\partial t^{\nu_{pq}}} \in C(D).$

If $K_{pq}(t,s) \equiv 0$, we set $\nu_{pq} = 0$. The IDAE (2) is called a ν-smoothing problem if $\mathscr{V}$ is a ν-smoothing operator.

We assume that the matrix kernel $K(t,s)$ of (2) does not vanish identically. Let $i \geq 0$ be an integer, $K^i, K_i, A_i, B_i \in \mathbb{R}^{d\times d}$, and denote by $(K^i)_{pq}$ and $(K_i)_{pq}$ as the element (p,q) of the matrix K^i and K_i, respectively. Let

$$K^0(t,s) := K(t,s),\ K_0 = K := K(t,t),$$
$$A_0 := A,\ B_0 := B - A_0P_0',\ A_1 := A_0 + B_0Q_0. \tag{4}$$

If $(K_i)_{pq}(t,t) \neq 0\,(i \geq 0)$, set $(K^{i+1})_{pq}(t,s) := 0$; otherwise

$$(K^{i+1})_{pq}(t,s) := \frac{\partial^{i+1}((K^i)_{pq}(t,s))}{\partial t^{i+1}}.$$