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OPTIMALITY CRITERIA AND DUALITY FOR NONLINEAR FRACTIONAL CONTINUOUS-TIME PROGRAMMING

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Abstract. This paper addressed the fractional continuous-time programming problem. The necessary and sufficient optimality conditions under generalized concavity assumptions are established. Dual problem to the primal one and investigate duality relations between them are also addressed.

Key words. Fractional programming, continuous-time programming, optimality conditions.

1. Introduction

Continuous-time programming problem originated from Bellman's "bottleneck" problems, considered in [2]. Since its first detailed introduction into the literature by Tyndall [15], many authors have contributed to the subject. Optimality criteria and duality theory, both in linear and nonlinear cases, have been investigated extensively. Nonlinear problem was first investigated, in 1968 by Hanson [6], Hanson and Mond [5] and in 1974 by Farr and Hanson [4]. Since then, a comprehensive bibliography has been produced. For more information, the reader refer to [3, 12, 20, 21, 22, 23].

In this paper, we consider fractional continuous-time problem. The problem of maximizing (or minimizing) the ratio of two real-valued functionals, subject to a set of constraints, is known as a fractional optimal control problem in the area of control theory. This class of problems is important for modeling various decision processes in the field of economics, game theory and operational research. They also often appear in some other contexts such as numerical analysis, approximation problems, facility location, optimal engineering design and information theory. For the reasons mentioned, fractional continuous-time programming problems have received major attention in the past thirty years, resulting into a comprehensive literature, dealing with their various theoretical and computational aspects. For recent results in this field, the reader is referred to [7, 8, 11, 13, 14, 16, 17, 18, 19]. In [13], optimality criteria is obtained for fractional continuous-time programming problem. Charnes-Cooper transformation, convexity and perturbation functions play a key role in these results. In papers [11, 19] fundamental tools were results given in [3, 23]. In [1], Arutyunov et al. have pointed out that such results in [3, 23] are not valid. Therefore, some results from aforementioned papers, unfortunately, are also not valid. Most recently, in [10], the authors have provided new necessary optimality conditions for nonlinear continuous-time programming problems with scalar valued objective function. Numerical algorithms for linear fractional continuous-time problem have been proposed in [16, 17]. In [16], the authors have introduced the discrete approximation method to solve the primal and dual pair of parametric continuous-time linear programming problems by using the recurrence method and provided numerical examples. The main purpose in [17] was to develop a discrete approximation method to solve a class of linear fractional continuous-time

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programming problems. Also, in mentioned paper, the authors have established an estimate of the error bound and have provided numerical examples to demonstrate the usefulness of this numerical algorithm.

Our aim in this paper is to provide necessary and sufficient optimality conditions for nonlinear fractional continuous-time problem. In addition, we shall construct duality model for the primal problem and prove suitable duality theorems.

The paper is organized in the following way. Some preliminaries about the problem are given in Section 2, where some important definitions are stated. In Section 3, necessary optimality conditions are obtained. Sufficient optimality conditions are obtained under concavity and generalized concavity assumptions. In Section 4 and 5, dual problems are presented and certain duality results are obtained.

2. Preliminaries

Consider the following fractional continuous-time programming problem:

$$\text{(FCTP)} \qquad \begin{aligned} &\max \quad \frac{\int_0^T f(t,x(t))\,dt}{\int_0^T g(t,x(t))\,dt} \\ &\text{s. t.} \quad h_i(t,x(t)) \geq 0, \quad i \in I = \{1,\ldots,m\} \text{ a.e. in } [0,T], \\ &\qquad\; x \in L_\infty([0,T];\mathbb{R}^n), \end{aligned}$$

where $f : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $g : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $h_i : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $i \in I$ are given functions. Here for each $t \in [0,T]$, $x_k(t)$ is the kth component of $x(t) \in \mathbb{R}^n$. All integrals are given in the Lebesgue sense. All vectors are column vectors. Inequality signs between vectors should be read componentwise. B denotes the open unit ball with centre at the origin, independently of the space or dimension. Let Ω_P be the set of feasible solutions of (FCTP),

$$\Omega_P = \{x \in L_\infty([0,T];\mathbb{R}^n) : h_i(t,x(t)) \geq 0, \quad i \in I, \text{ a.e. in } [0,T]\}.$$

Let $\varepsilon > 0$ and $\hat{x} \in \Omega_P$. Suppose the following assumptions are valid:

(i) Functions $f(t,\cdot)$ and $g(t,\cdot)$ are continuously differentiable on $\hat{x}(t)+\varepsilon\bar{B}$ a.e. in $[0,T]$. Functions $f(t,\cdot)$ and $g(t,\cdot)$ are Lebesgue measurable for each x, and there exist numbers $K_f > 0$ and $K_g > 0$ such that

$$\|\nabla f(t,\hat{x}(t))\| \leq K_f \text{ a.e in } [0,T],$$

$$\|\nabla g(t,\hat{x}(t))\| \leq K_g \text{ a.e in } [0,T];$$

(ii) For each $i \in I$, the function $h_i(t,\cdot)$ is continuously differentiable on $\hat{x}(t)+\varepsilon\bar{B}$ a.e. in $[0,T]$. For each $i \in I$, the function $h_i(\cdot,x(\cdot))$ is essentially bounded in $[0,T]$ for all $x \in L_\infty([0,T],\mathbb{R}^n)$ and there exists a number $K_h > 0$ such that

$$\|\nabla h_i(t,\hat{x}(t))\| \leq K_h,\ i \in I, \text{ a.e in } [0,T].$$

For $x \in \Omega_P$, we also assume that

$$\int_0^T f(t,x(t))\,dt \geq 0 \quad \text{and} \quad \int_0^T g(t,x(t))\,dt > 0. \tag{1}$$