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Sublinear Operators with Rough Kernel on Herz Spaces with Variable Exponents

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Abstract. We prove some boundedness results for a large class of sublinear operators with rough kernel on the homogeneous Herz spaces where the three main indices are variable exponents. Some known results are extended.

Key Words: Herz space, variable exponent, sublinear operator.

AMS Subject Classifications: 42B35, 46E30

1 Introduction

Suppose that $\mathbb{S}^{n-1}$ is the unit sphere of $\mathbb{R}^n$ $(n \geq 2)$ equipped with normalized Lebesgue measure $d\sigma(x')$. Let $\Omega \in L^1(\mathbb{S}^{n-1})$ be homogeneous of degree zero and satisfy

$$\int_{\mathbb{S}^{n-1}} \Omega(x')d\sigma(x') = 0,$$

where $x' = x/|x|$ for any $x \neq 0$. In this paper, we will consider sublinear operators which satisfy that for any $f \in L^1(\mathbb{R}^n)$ with compact support and $x \notin \mathrm{supp} f$,

$$|T_\Omega f(x)| \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^n}|f(y)|dy, \tag{1.1}$$

and their corresponding fractional versions

$$|T_{\Omega,\beta} f(x)| \leq C \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-\beta}}|f(y)|dy, \tag{1.2}$$

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where $0<\beta<n$ and $C>0$ is an absolute constant.

Soria and Weiss [20] first introduced the condition (1.1), which is satisfied by many classical operators in harmonic analysis, such as the Calderón-Zygmund operators, Carleson's maximal operators, Hardy-Littlewood maximal operators, etc. In the case $\Omega \in L^s(\mathbb{S}^{n-1})$ for some $s\in[1,\infty]$, Lu et al. [15] proved the boundedness of sublinear operators T_Ω and $T_{\Omega,\beta}$ on generalized Morrey spaces. Hu et al. [10] established the boundedness of sublinear operators with rough kernel on the classical Herz spaces. We refer to [13] for further results on these operators.

In recent years, function spaces with variable exponents have attracted more and more attention. The growing interest in such spaces is strongly stimulated by the treatment of recent problems in fluid dynamics [19], image restoration [3] and PDE with non-standard growth conditions [7]. The generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$ (also known as Lebesgue spaces with variable exponent) and the corresponding generalized Sobolev spaces $W^{k,p(\cdot)}(\mathbb{R}^n)$ have been systematically studied by Kováčik and Rákosník in [12]. Since then various other function spaces such as Herz spaces [11], Morrey type spaces [8,16] and so on have been investigated in the variable exponent setting.

As shown in [14,18], Herz spaces play a crucial role in harmonic analysis and PDE. For instance, they appear in the characterization of multiplier on Hardy spaces and in the regularity theory for elliptic and parabolic equations in divergence form. Herz spaces $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ and $K^{\alpha(\cdot)}_{p(\cdot),q}(\mathbb{R}^n)$ with variable exponent α, p but fixed $q\in\mathbb{R}$ were first studied by Almeida and Drihem [1], and they also studied the boundedness of a wide class of sublinear operators on these spaces. Recently, Drihem and Seghiri in [6] generalized some of the main results in [1] to the Herz spaces $\dot{K}^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $K^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, where the exponent q is variable as well. The main purpose of this paper is to further extend these results to the rough kernel case.

In general, by B we denote the ball with center $x\in\mathbb{R}^n$ and radius $r>0$. If E is a subset of $\mathbb{R}^n$, $|E|$ denotes its Lebesgue measure and χ_E its characteristic function. p' denotes the conjugate exponent defined by $\frac{1}{p}+\frac{1}{p'}=1$. We use $x\approx y$ if there exist constants c_1, c_2 such that $c_1x\le y\le c_2x$. The symbol C stands for a positive constant, which may vary from line to line.

2 Preliminaries and lemmas

We begin with a brief and necessarily incomplete review of the variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$, see [4,5] for more information.

Let $\mathcal{P}(\mathbb{R}^n)$ denote the set of all measurable functions $p(\cdot):\mathbb{R}^n\to[1,\infty)$. For $p(\cdot)\in\mathcal{P}(\mathbb{R}^n)$, we use the notation

$$p_- := \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad p_+ := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x).$$