

A Conservative and Monotone Characteristic Finite Element Solver for Three-Dimensional Transport and Incompressible Navier-Stokes Equations on Unstructured Grids

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Abstract. We propose a mass-conservative and monotonicity-preserving characteristic finite element method for solving three-dimensional transport and incompressible Navier-Stokes equations on unstructured grids. The main idea in the proposed algorithm consists of combining a mass-conservative and monotonicity-preserving modified method of characteristics for the time integration with a mixed finite element method for the space discretization. This class of computational solvers benefits from the geometrical flexibility of the finite elements and the strong stability of the modified method of characteristics to accurately solve convection-dominated flows using time steps larger than its Eulerian counterparts. In the current study, we implement three-dimensional limiters to convert the proposed solver to a fully mass-conservative and essentially monotonicity-preserving method in addition of a low computational cost. The key idea lies on using quadratic and linear basis functions of the mesh element where the departure point is localized in the interpolation procedures. The proposed method is applied to well-established problems for transport and incompressible Navier-Stokes equations in three space dimensions. The numerical results illustrate the performance of the proposed solver and support its ability to yield accurate and efficient numerical solutions for three-dimensional convection-dominated flow problems on unstructured tetrahedral meshes.

AMS subject classifications: 65M25, 65N30, 65Z05, 35Q30

Key words: Mass-conservative, monotonicity-preserving, modified method of characteristics, finite element method, convection-dominated problems, incompressible Navier-Stokes equations.

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1 Introduction

Transport in incompressible flows takes place in many applications in science and engineering. This class of problems occur in a many applications in nature and technology, for example in the simulation of a heat transport in draining films [45], groundwater flows in soils [25], and the transport of ferro-fluids under magnetic fields [4] among others. Developing robust numerical solvers for this set of problems is still challenging in the situation of convection-dominated flows for which convection terms are manifestly more important than the diffusion terms particularly if some nondimensional parameters attend high values. As example of these parameters we mention the well-known Reynolds number for the incompressible Navier-Stokes equations and the Peclet number for the convection-diffusion equations. There exist many numerical techniques in the literature to solve the transport and incompressible Navier-Stokes equation. In case of convection-dominated flows, the conventional Eulerian finite element methods use up-stream weighting in their implementations to stabilize the discretization. For example, the most popular Eulerian finite element methods are the streamline upwind Petrov-Galerkin methods [3, 5, 11], the Taylor-Galerkin methods [8, 12, 16] and the Galerkin/least-squares methods [3, 9, 26]. However, truncation errors generated by the time integration in these conventional Eulerian methods are dominant and require Courant-Friedrichs-Lewy (CFL) stability conditions which impose sever restrictions on the time steps used in the numerical computations. Eulerian numerical methods for three-dimensional advection-diffusion problems have also been investigated in [10, 13, 27, 35, 48] among others. In [13], a simple comparison between implicit and explicit finite difference methods have been studied for a class of linear three-dimensional advection-diffusion problems with constant coefficients. However, all results presented were in Cartesian meshes which restrict their application to simple regular domains. High-order compact finite difference methods have also been proposed in [48] for the stationary semi-linear three-dimensional advection-diffusion equations. Eulerian-based methods for the three-dimensional incompressible Navier-Stokes equations have also been discussed in [10, 14, 15, 32, 33, 36, 41, 43, 44] among others. A dimension split method has been studied in [10] and a multi-stage Rosenbrock scheme has been applied to the three-dimensional incompressible Navier-Stokes equations in [14]. However, these methods fail to resolve flow structures at high Reynolds numbers. In [44], a multigrid adaptive unstructured finite element method has been proposed for the numerical solution of the three-dimensional incompressible Navier-Stokes equations. However, the adaptation process in this scheme requires assembling matrices at each time step which increases the computational cost. A compact mixed finite element method has been proposed in [43] to reduce the computing time for solving linear algebraic equations resulted from the discretization of three-dimensional incompressible Navier-Stokes equations but this study dealt with steady problems only. In [32, 33], a class of finite difference schemes have been implemented for space discretization of three-dimensional incompressible Navier-Stokes equations. However, the main drawback of these methods is that they are not able to resolve complex flow problems in irregular