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A New Mapped WENO Scheme Using Order-Preserving Mapping

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Abstract. Existing mapped WENO schemes can hardly prevent spurious oscillations while preserving high resolutions at long output times. We reveal in this paper the essential reason of such phenomena. It is actually caused by that the mapping function in these schemes can not preserve the order of the nonlinear weights of the stencils. The nonlinear weights may be increased for non-smooth stencils and be decreased for smooth stencils. It is then indicated to require the set of mapping functions to be *order-preserving* in mapped WENO schemes. Therefore, we propose a new mapped WENO scheme with a set of mapping functions to be order-preserving which exhibits a remarkable advantage over the mapped WENO schemes in references. For long output time simulations of the one-dimensional linear advection equation, the new scheme has the capacity to attain high resolutions and avoid spurious oscillations near discontinuities meanwhile. In addition, for the two-dimensional Euler problems with strong shock waves, the new scheme can significantly reduce the numerical oscillations.

AMS subject classifications: 65M06, 65M12

Key words: Mapped WENO, order-preserving mapping, hyperbolic problems.

1 Introduction

Many numerical methods have been studied to solve the hyperbolic problems that may generate discontinuities as time evolves in its solution even if the initial condition is smooth, especially for nonlinear cases. As the discontinuities often cause spurious oscillations in numerical calculations, it is very difficult to design high order numerical schemes. Thus, the essentially non-oscillatory (ENO) schemes [1–4] and weighted ENO

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(WENO) schemes [5–9] have developed quite successfully in recent decades, and they are very popular methods to solve the hyperbolic conservation laws because of their success to the ENO property. The goal of this paper is to devise a new version of the fifth-order finite volume WENO scheme for solving the following hyperbolic conservation laws

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t}+\displaystyle\sum_{\alpha=1}^{d}\dfrac{\partial \mathbf{f}_\alpha(\mathbf{u})}{\partial x_\alpha}=0, \quad x_\alpha\in\mathbb{R}, \quad t>0, \\ \mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), \end{cases} \tag{1.1}$$

with proper boundary conditions. Here, $\mathbf{u}=(u_1,\cdots,u_m)\in\mathbb{R}^m$ are the conserved variables and $\mathbf{f}_\alpha:\mathbb{R}^m\rightarrow\mathbb{R}^m$, $\alpha=1,2,\cdots,d$ are the Cartesian components of flux.

The first WENO scheme was developed by Liu et al. [7] in the finite volume version. It converts an rth-order ENO scheme [1–3] into an $(r+1)$th-order WENO scheme through a convex combination of all candidate substencils instead of only using the optimal smooth candidate stencil by the original ENO scheme. By introducing a different definition of the smoothness indicators used to measure the smoothness of the numerical solutions on substencils, the classic $(2r-1)$th-order WENO-JS scheme was proposed by Jiang and Shu [8]. The WENO-JS scheme has been successfully used in a wide number of applications. Later, more versions of the smoothness indicators have been devised and used successfully [30–33]. The WENO methodology within the general framework of smoothness indicators and nonlinear weights proposed in the WENO-JS scheme [8] is still in development to improve the convergence rate in smooth regions [10–15] and reduce the computational cost [16–20].

Henrick et al. [10] pointed out that the classic WENO-JS scheme was less than fifth-order for many cases such as at or near critical points of order $n_{\text{cp}}=1$ in the smooth regions. Here, we refer to n_{cp} as the order of the critical point; e.g., $n_{\text{cp}}=1$ corresponds to $f'=0$, $f''\neq 0$ and $n_{\text{cp}}=2$ corresponds to $f'=0$, $f''=0$, $f'''\neq 0$, etc. The necessary and sufficient conditions on the nonlinear weights for optimality of the convergence rate of the fifth-order WENO schemes were derived by Henrick et al. in [10]. These conditions were reduced to a simple sufficient condition [16] which could be easily extended to the $(2r-1)$th-order WENO schemes [11]. Then, by designing a mapping function which satisfies the sufficient condition to achieve the optimal order of accuracy, the original mapped WENO scheme, named WENO-M, was devised by Henrick et al. [10].

Recently, Feng et al. [12] noted that, when the WENO-M scheme was used for solving the problems with discontinuities for long output times, its mapping function may amplify the effect from the non-smooth stencils leading to a potential loss of accuracy near discontinuities. To address this issue, they proposed a piecewise polynomial mapping function with two additional requirements, that is, $g'(0)=0$ and $g'(1)=0$ ($g(x)$ denotes the mapping function), to the original criteria in [10]. However, the resultant WENO-PMk scheme [12] may generate the non-physical oscillations near the discontinuities as shown in Fig. 8 of [11] and Figs. 3-8 of [21]. Later, Feng et al. [11] devised an improved mapping method which is referred to as WENO-IM(k,A) where k is a positive even integer and