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Backward and Forward Modified SOR Iteration Methods for Solving Standard Saddle-Point Problems

Fang Chen[1,*], Bi-Cong Ren[1] and Galina V. Muratova[2]

[1] *School of Applied Science, Beijing Information Science and Technology University, Beijing 100192, P.R. China.*
[2] *Laboratory of Computational Mechanics, I.I. Vorovich Institute of Mathematics, Mechanics and Computer Science, Southern Federal University, Rostov-on-Don 344090, Russia.*

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Abstract. For the standard saddle-point problem, we discuss and analyze the backward and forward modified successive overrelaxation (SOR) iteration methods in detail. Theoretical analysis shows that when the parameters are selected appropriately, the backward and forward modified SOR iteration methods converge to the unique solution of the standard saddle-point problem. Finally, numerical examples show the effectiveness of our iteration methods.

AMS subject classifications: 65F08, 65F10, CR:G1.3

Key words: Standard saddle-point problem, iteration method, convergence.

1. Introduction

We consider the standard saddle-point problem

$$\mathscr{A} z \equiv \begin{pmatrix} A & B \\ -B^T & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} \equiv b, \tag{1.1}$$

where $x, p \in \mathbb{R}^m$, $y, q \in \mathbb{R}^n$, $A \in \mathbb{R}^{m\times m}$ is a nonsymmetric positive definite matrix, that is, the matrix $H = (A + A^T)/2$, the symmetric part of A, is positive definite, and $B \in \mathbb{R}^{m\times n}$ is a matrix of full column rank with $m \geq n$. Here, B^T denotes the transpose of the matrix of B. The standard saddle-point problem (1.1) arises in many fields of scientific computing and engineering applications such as the finite element discretization of partial differential equations including the Stokes and Navier-Stokes equations [22], the constrained and

*Corresponding author. *Email addresses:* `chenfreesky@126.com` (F. Chen), `study_rbc@163.com` (B.-C. Ren), `muratova@sfedu.ru` (G.V. Muratova)

generalized least-squares problems [30], the constrained optimization [24], the fluid dynamics [21,25], the mixed finite element method for elliptic equation [28] (see [4,11] for more details).

There are various iteration methods for solving the standard saddle-point problem (1.1). In recent years, many researchers have paid attention to the matrix splitting iteration methods for solving the standard saddle-point problem. There are various methods based on the Uzawa, SOR and HSS iteration methods. For the Uwaza-type methods, Bai and Wang [15] presented the parameterized inexact Uzawa method to solve the generalized saddle-point problem. About the SOR-type methods, Golub *et al.* [23] proposed an SOR-like iteration method for solving the standard saddle-point problem (1.1). Bai *et al.* [12] presented the generalized successive overrelaxation (GSOR) and the generalized accelerated overrelaxation (GAOR) iteration methods to solve the standard saddle-point problem (1.1). Bai *et al.* [16] proposed the shift-splitting (SS) iteration method and SS preconditioner to solve the non-Hermitian positive definite linear systems, and later, Cao *et al.* [18] used the SS idea to solve the standard saddle-point problem (1.1). In addition, Bai *et al.* [9] presented the Hermitian and skew-Hermitian splitting (HSS) iteration method in 2003 to solve the non-Hermitian positive definite linear systems, and later, Benzi and Golub [17] used the HSS iteration method to solve the saddle-point problem. For more references see [7,19,20,26,27,29,32].

In this paper, we establish a class of backward and forward modified successive overrelaxation (BMSOR and FMSOR) iteration methods to solve the standard saddle-point problem (1.1). These two block relaxation iteration methods are indeed specialized single-step iteration cases of the SSOR-like iteration method proposed and analyzed in [6], but now they are used for solving the standard saddle-point problem (1.1) instead of a general nonsingular linear system.

The rest of the paper is organized as follows. In Section 2, we develop BMSOR iteration method for solving the standard saddle-point problem (1.1), discuss the convergence of BMSOR iteration method, and analyze the convergence domain of the involved parameters. In Section 3, the FMSOR iteration method is proposed. Furthermore, we derive the convergence property of the FMSOR iteration method for solving the standard saddle-point problem (1.1). Numerical experiments are performed to show the effectiveness of the BMSOR and FMSOR iteration methods in Section 4. Finally, the paper is concluded in Section 5.

2. BMSOR Iteration Method and Its Convergence

In this section, we propose the BMSOR iteration method for solving the standard saddle-point problem (1.1).

First of all, we assume that the matrix $A \in \mathbb{R}^{m\times m}$ is nonsymmetric and positive definite, and the matrix $B \in \mathbb{R}^{m\times n}$ has full column rank. Let $H = (A + A^T)/2$ and $S = (A - A^T)/2$ be the symmetric and skew-symmetric parts of A. We make the following matrix splitting for the coefficient matrix $\mathscr{A}$ of the standard saddle-point problem (1.1):