

High Order Finite Difference WENO Methods with Unequal-Sized Sub-Stencils for the Degasperis-Procesi Type Equations

Jianfang Lin¹, Yan Xu², Huiwen Xue¹ and Xinghui Zhong^{1,*}

¹ School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310058, P.R. China.

² School of Mathematical Sciences, University of Science and Technology of China, Hefei, Anhui 230026, P.R. China.

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Abstract. In this paper, we develop two finite difference weighted essentially non-oscillatory (WENO) schemes with unequal-sized sub-stencils for solving the Degasperis-Procesi (DP) and μ -Degasperis-Procesi (μ DP) equations, which contain nonlinear high order derivatives, and possibly peakon solutions or shock waves. By introducing auxiliary variable(s), we rewrite the DP equation as a hyperbolic-elliptic system, and the μ DP equation as a first order system. Then we choose a linear finite difference scheme with suitable order of accuracy for the auxiliary variable(s), and two finite difference WENO schemes with unequal-sized sub-stencils for the primal variable. One WENO scheme uses one large stencil and several smaller stencils, and the other WENO scheme is based on the multi-resolution framework which uses a series of unequal-sized hierarchical central stencils. Comparing with the classical WENO scheme which uses several small stencils of the same size to make up a big stencil, both WENO schemes with unequal-sized sub-stencils are simple in the choice of the stencil and enjoy the freedom of arbitrary positive linear weights. Another advantage is that the final reconstructed polynomial on the target cell is a polynomial of the same degree as the polynomial over the big stencil, while the classical finite difference WENO reconstruction can only be obtained for specific points inside the target interval. Numerical tests are provided to demonstrate the high order accuracy and non-oscillatory properties of the proposed schemes.

AMS subject classifications: 65M06, 35Q53

Key words: High order accuracy, weighted essentially non-oscillatory schemes, Degasperis-Procesi equation, μ -Degasperis-Procesi equation, finite difference method, multi-resolution.

*Corresponding author. *Email addresses:* jianfang.lin@zju.edu.cn (J. Lin), yxu@ustc.edu.cn (Y. Xu), xuehuiwen1105@163.com (H. Xue), zhongxh@zju.edu.cn (X. Zhong)

1 Introduction

In this paper, we are interested in solving the Degasperis-Procesi (DP) equation

$$u_t - u_{txx} + 4f(u)_x = f(u)_{xxx}, \quad (1.1)$$

with $x \in \Omega \subset \mathbb{R}$ and $f(u) = u^2/2$, and the μ -Degasperis-Procesi (μ DP) equation

$$\mu(u)_t - u_{txx} + 3\mu(u)u_x = 3u_x u_{xx} + uu_{xxx}, \quad (1.2)$$

where $x \in \mathcal{S}^1 = \mathbb{R}/\mathbb{Z}$ (the circle whose perimeter equals 1), and $\mu(u) = \int_{\mathcal{S}^1} u dx$ denotes the mean of u on \mathcal{S}^1 . We develop two finite difference weighted essentially non-oscillatory (WENO) schemes for solving (1.1) and (1.2) with unequal-sized sub-stencils, which provide a simpler way for the reconstruction procedure than the classical WENO schemes, while still simultaneously maintaining high order accuracy in smooth regions and controlling spurious numerical oscillations near discontinuities.

The DP equation was singled out first in [11] by an asymptotic integrability test within a family of third order dispersive equations in the form of

$$u_t + c_0 u_x + \gamma u_{xxx} - \alpha^2 u_{txx} = (c_1 u^2 + c_2 u_x^2 + c_3 uu_{xx})_x, \quad (1.3)$$

with $\gamma, \alpha, c_0, c_1, c_2$ and c_3 being real constants. The DP equation (1.1) can be transformed from (1.3) with $c_1 = -\frac{2c_3}{\alpha^2}$, $c_2 = c_3$, see [10] for more details. It is one of the only three equations that satisfy the asymptotic integrability condition, besides the Korteweg-De Vries (KdV) equation ($\alpha = c_2 = c_3 = 0$) and the Camassa-Holm (CH) equation ($c_1 = -\frac{3c_3}{2\alpha^2}$, $c_2 = \frac{c_3}{2}$). The DP equation can be regarded as an approximate model of shallow water wave propagation in small amplitude and long wavelength regime [9, 13, 17, 20], and its asymptotic accuracy is the same as the CH equation (one order more accurate than the KdV equation). The well-posedness of the DP equation has been studied in [6–8, 38–41] and the cited references therein. The μ DP equation is an extensive study of the DP equation. It can be regarded as an evolution equation on the space of tensor densities over the Lie algebra of smooth vector fields on the circle.

One of the important features of the DP type equations is that they admit not only peakon solutions [10], but also shock waves [6, 27]. Explicit expressions of multi-peakon and multi-shock solutions were provided in [27–29] for the DP equation, and in [21] for the μ DP equation. Another feature of the DP type equations is that they satisfy those conservation laws which cannot guarantee the bound of the H^1 -norm of the solution. Due to these features, it is very difficult to design stable and high order accurate numerical methods for solving the DP and μ DP equations. For the DP equation, the existing numerical methods include the particle method based on the multi-shock peakon solution [16], operator splitting finite difference methods [8, 14], local discontinuous Galerkin (DG) methods [37], conservative finite difference methods [30], compact finite difference methods with symplectic implicit Runge-Kutta (RK) time integration [42], direct DG methods [24], and Fourier spectral methods [3, 35], etc. Local DG method was developed for the μ DP