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RECOVERY-BASED A POSTERIORI ERROR ESTIMATION FOR ELLIPTIC INTERFACE PROBLEMS BASED ON PARTIALLY PENALIZED IMMERSED FINITE ELEMENT METHODS

YANPING CHEN, ZHIROU DENG, AND YUNQING HUANG

Abstract. This paper develops a recovery-based a posteriori error estimation for elliptic interface problems based on partially penalized immersed finite element (PPIFE) methods. Due to the low regularity of solution at the interface, standard gradient recovery methods cannot obtain superconvergent results. To overcome this drawback a new gradient recovery method is proposed that applies superconvergent cluster recovery (SCR) operator on each subdomain and weighted average (WA) operator at recovering points on the approximated interface. We prove that the recovered gradient superconverges to the exact gradient at the rate of $O(h^{1.5})$. Consequently, the proposed method gives an asymptotically exact a posteriori error estimator for the PPIFE methods and the adaptive algorithm. Numerical examples show that the error estimator and the corresponding adaptive algorithm are both reliable and efficient.

Key words. Interface problems, a posteriori error estimation, immersed finite element methods, gradient recovery, adaptive algorithm.

1. Introduction

Interface problems arise in many applications of fluid mechanics and material science, where its governing partial differential equations (PDEs) have discontinuous coefficients. Solving interface problems accurately and efficiently is a challenge because these coefficients across the interface lead to low global regularity of the solution. Finite element (FE) methods for interface problems have been widely studied, which can be roughly divided into two categories: interface-fitted FE methods [1–4] and interface-unfitted FE methods [5–7]. The interface-fitted FE methods require the partition to be aligned with the interface. It is well known that these methods have the optimal convergence rate in both L^2 and energy norms [8, 9] when the exact solution has sufficient regularity. However, it is usually difficult and time-consuming to generate the required body-fitted meshes, especially when the interface geometry is complex. There are increasing interests in developing interface-unfitted FE methods using interface-unfitted meshes. The immersed finite element (IFE) methods proposed by Li et al. [10,11] are based on finite element discretizations on interface-unfitted meshes. The main idea of IFE methods is to construct basis functions satisfying the jump conditions on the interface elements to obtain sharp solutions around the interface. In [12], Li et al. proved that the IFE space has the optimal approximation capability. However, Chou et al. showed in [13] that the non-conforming IFE methods do not have fully second order accurate. It is second order in the L^2 norm, but only first order convergence in the L^∞ norm due to the consistent error resulted from the discontinuities of test functions. Some improved IFE methods have been developed to eliminate the consistent error,

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for instance, symmetric and consistent immersed finite element (SCIFE) methods [14], Petrov-Galerkin immersed finite element (PGIFE) methods [15] and partially penalized immersed finite element (PPIFE) methods [6] and so on.

Compared with the standard IFE methods, PPIFE methods introduce extra stabilization terms only at the interface edges for penalizing the inter-element discontinuity. The optimal convergence rate is theoretically proved for the PPIFE methods in the energy norm and L^2 norm [6, 16]. Moreover, the PPIFE methods were extended to the semi-linear elliptic interface problems [17], the second-order elliptic interface problems with non-homogeneous jump conditions in [18], and other types of interface problems in [19, 20].

Error analysis is a classic topic in FE methods, and it is typically categorized into a priori and a posteriori error estimates. The a priori error estimates of IFE methods have been well developed in the past decade, but the a posteriori error estimates of IFE methods are still in the initial stage. Cao et al. [21] introduced a new approach for constructing IFE basis functions based on the theory of orthogonal polynomials. They proved that IFE methods for one dimensional general elliptic interface problems have nodal superconvergence at the roots of generalized orthogonal polynomials. For two dimensional cases, Guo et al. [22] designed an improved polynomial preserving recovery (IPPR) method for SCIFE and PGIFE methods, and verified its superconvergence by numerical examples. In [23], they also showed the supercloseness results for PPIFE methods and proved that the recovered gradient using the IPPR operator is superconvergent to the exact gradient. He et al. [24] proposed and analyzed the residual-based a posteriori error estimation of the PPIFE methods for solving elliptic interface problems. The a posteriori error estimate is proved to be reliable and efficient.

The purpose of this paper is to develop and analyze a novel recovery-based a posteriori error estimation of the PPIFE methods for elliptic interface problems. Standard gradient recovery methods, including superconvergent patch recovery (SPR) [25, 26], polynomial preserving recovery (PPR) [27–29] and superconvergent cluster recovery (SCR) [30], cannot obtain the superconvergent recovered gradient due to the low regularity of numerical solution at the interface. A recovery-based a posteriori error estimator based on these methods will result in over-refinement as studied in [31]. Therefore, these methods cannot be directly applied to the interface problems. Note that the SCR method obtains the recovered gradient at recovering points by taking derivatives of a linear polynomial, in which the linear polynomial is acquired by least-square fitting the solution values in a cluster of sampling points. This recovery procedure is simpler and effective, and maintains the superconvergence properties of SPR and PPR methods [30]. Hence, our gradient recovery method is based on the SCR method.

In this paper, we propose a new gradient recovery operator for PPIFE methods, and prove that the operator is linear, bounded, and consistent. The establishment of the operator relies on three observations: (i) IFE solution is piecewise smooth on each subdomain, although it has low global regularity; (ii) the solution is discontinuous at the interface, even if the exact solution is continuous; (iii) the SCR operator is local. Accordingly, the gradient recovery operator is constructed by two steps: enriching and smoothing. We first design an enriching operator to make the discontinuous FE solution continuous on a local body-fitted mesh generated by adding extra nodes [10], and then apply the SCR operator to the enriched FE