JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **35**, No. 2, pp. 123-147

doi: 10.4208/jpde.v35.n2.2
June 2022

Dynamics for Three Dimensional Generalized Navier-Stokes Equations with Delay

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Received 14 May 2021; Accepted 4 August 2021

Abstract. This paper is concerned with the existence of pullback attractors for three dimensional generalized Navier-Stokes equations with delay. According to compact argument, the existence and uniqueness of weak solutions are proved by using Galerkin method, and the continuous dependence of solutions on initial values is also shown. Based on the asymptotic compactness via weak convergence method and pullback absorbing set on appropriate functional phase spaces, we get the existence of pullback attractors.

AMS Subject Classifications: 35B40, 35Q35

Chinese Library Classifications: O175.2

Key Words: Three dimensional generalized Navier-Stokes equations; delay; pullback attractor.

1 Introduction

As early as the end of the 18th century, it has been found that the change rule of many things is not only dependent on the current state, but also related to the historical state. In this case, we need to consider the influence of time delay. Later on, the delay differential equation is used to describe the development system that depends on both the current state and the past state. Its characteristic is to fully consider the influence of the history of

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the system on status quo. Its general form is functional differential equation, which has been studied for more than 200 years. In 1771, Condorcet [1] derived the first functional differential equation in history, but the systematic research work did not really appear until the 1950s. By the 1970s, the research on delay differential equation theory has been developed greatly. Important achievements have been made in many aspects such as the basic theory of solution and stability theory, etc., see literature [1–7], among which the reference [3] is a good summary of the bounded delay functional differential equation research. Moreover, the delay differential equation can be applied to many practical problems, such as studying the law of single population growth, ship stability control, etc., which has important research significance.

The study of Navier-Stokes equations is important for understanding fluid turbulence and can generally be used to describe the motion of gases and liquids. For a long time, Navier-Stokes equations have been concerned and studied by many scholars such as Ladyzhenskaya, Temam, Robinson, Caraballo, Real and so on. For their related articles and books, see references [8–17]. There are also many studies on the Navier-Stokes equations with time delay terms. The first one was initiated by Caraballo and Real [15] and many important results were obtained [18–22]. Among them, the references [18, 19, 21] studied the two-dimensional Navier-Stokes equations with time delay, while literature [20, 22] studied the three-dimensional modified Navier-Stokes equations with time delay.

For the classical three-dimensional Navier-Stokes equation, the uniqueness of its weak solution is unknown, which makes it difficult to study its asymptotic behavior. Lions [23] and Prodi [24] studied the existence and uniqueness of weak solutions for three-dimensional Navier-Stokes equations with nonlinear viscosity. Caraballo and Real [16] gave a general theorem for the existence and uniqueness of solutions of two-dimensional Navier-Stokes equations with time delays and for the existence of solutions in three-dimensional cases. Caraballo and Han [25] studied stability of stationary solutions to 2D-Navier-Stokes models with delays. Quyet [26] studied the existence of pullback attractor for two-dimensional g-Navier-Stokes equations with infinite delay. Caraballo and Han [27] studied the existence, uniqueness and asymptotic properties of solutions of Navier-Stokes models with time delays. The results of the existence and upper semicontinuity of the attractor for time-delay equations can be referred to [28–32]. Yang and his collaborators studied the structure and stability of the pullback attractor for the three-dimensional Brinkman-Forchheimer equation with time delay in [33]. As for the existence results of the attractor for time-delay equations can be referred to [34–37], in which the studies on random time-delay equations are mainly referred to [35–37]. However, as our best knowledge, there is less results for the well-posedness and pullback dynamics for three dimensional incompressible generalized Navier-Stokes equations, which attracts our attention.

This paper is concerned with the pullback dynamics for the following three-dimensio-