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A COMPUTATIONAL STUDY OF PRECONDITIONING TECHNIQUES FOR THE STOCHASTIC DIFFUSION EQUATION WITH LOGNORMAL COEFFICIENT

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Abstract. We present a computational study of several preconditioning techniques for the GMRES algorithm applied to the stochastic diffusion equation with a lognormal coefficient discretized with the stochastic Galerkin method. The clear block structure of the system matrix arising from this type of discretization motivates the analysis of preconditioners designed according to a field-splitting strategy of the stochastic variables. This approach is inspired by a similar procedure used within the framework of physics based preconditioners for deterministic problems, and its application to stochastic PDEs represents the main novelty of this work. Our numerical investigation highlights the superior properties of the field-split type preconditioners over other existing strategies in terms of computational time and stochastic parameter dependence.

Key words. Stochastic diffusion equation, lognormal coefficient, stochastic Galerkin method, field-split, preconditioning, geometric multigrid, GMRES.

1. Introduction

In the last decade, stochastic partial differential equations (SPDEs) have attracted great attention from the scientific community, due to their ability to take into account uncertainties entering the problem through the input data. These sources of uncertainty may arise for instance from boundary and initial conditions, coefficients, forcing terms, or intrinsic randomness of the processes as in the case of heterogeneous media [1, 2, 3, 4, 5, 6]. The solutions of SPDEs allows to characterize the mean, variance and in general the probability density function of quantities of interest in the post-processing phase [7, 8, 9]. Among a wide variety of numerical methods for solving SPDEs [10], a popular approach is the stochastic Galerkin method (SGM), where a Galerkin projection is employed to approximate the infinite dimensional stochastic space with a finite dimensional one, spanned by appropriate basis functions [11].

In this work, we focus on the efficient numerical solution of the stochastic diffusion equation with a lognormal coefficient, arising for instance in the framework of groundwater flows, where the permeability coefficient is often considered to be lognormal [12, 13]. Using the SGM with a polynomial chaos expansion of the exponential coefficient, the resulting stiffness matrix is block dense, due to the nonlinearity of the coefficient [14], and ill-conditioned with respect to the mesh size and to the stochastic parameters such as the standard deviation of the input lognormal field [15, 16, 17, 18]. The aforementioned properties of the matrix require the design of ad hoc preconditioned solvers for the efficient solution of the SGM system. When the stochastic diffusion problem is well-posed, regardless of the type of coefficient projected onto the stochastic basis, the SGM system matrix is symmetric and positive definite, hence a huge variety of preconditioned conjugate gradient (PCG) solvers has been proposed in the literature [19, 20, 21, 22, 15, 23, 9, 24]. As pointed out in [14] however, the density of the matrix given by a lognormal

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coefficient makes the use of PCG methods problematic, given that for every PCG iteration a matvec operation has to be performed. On the other hand, in two recent studies very relevant to our framework [15, 14], the preconditioned GMRES algorithm has been shown to perform better than PCG in terms of both solution time and dependence on the stochastic parameters for a stochastic diffusion equation with lognormal coefficient. In the above mentioned studies, the diffusion problem is reformulated as a convection-diffusion problem with the result that the nonlinear coefficient is transformed into a linear one, and the resulting matrix gains sparsity, while losing symmetry. The gain in sparsity is due to the fact that the system matrix is obtained from the discretization of a stochastically linear problem, hence the result is a block sparse matrix with at most $2N+1$ non zero blocks per row [15], with N being the number of stochastic variables. The preconditioned GMRES from [15] and [14] showed independence on spatial discretization and most stochastic parameters, however a mild but relevant influence on the standard deviation of the input lognormal field was observed (with the number of iterations going from 6 to 27 in the worst case scenario in [15]). Hence, the important work done in the two studies described above motivated our choice of studying the performances of a preconditioned GMRES algorithm rather than a PCG as most studies in the literature have done. We also mention [25] for a work that has employed a flexible GMRES algorithm as a solver.

After choosing the solver, the next crucial step is the choice of appropriate preconditioners. Several different strategies have been carried out for this task, with the most popular possibly being the so called mean-based preconditioner [26, 9]. Variants of the mean-based preconditioner have also been designed [18]. Moreover, domain decomposition type methods have also been used as preconditioning techniques [21, 19, 25], as well as low-rank approaches [20] and other approaches such as multigrid [23, 24]. In particular, the extensive study on iterative solvers for stochastic PDEs carried out in [23] reported that for the case of a lognormal field, only a CG solver preconditioned with multigrid with a block Gauss-Seidel smoother showed robust convergence, i.e. close to steady number of iterations, as the other approaches considered suffered a dependence on the stochastic parameters, such as the variance of the input field. The study in [23] also concluded that for large problems, multigrid type methods should be preferred. Hence, in light of [23], in the present study we decided to consider preconditioners of multigrid type, specifically geometric multigrid. For completeness, we also mention works that employed geometric or algebraic multigrid as a solver, although not in the context of a lognormal diffusion coefficient [1, 27].

The major novelty of this work is the introduction of block preconditioners for the multigrid smoother with a structure arising from using the stochastic modes as a field-splitting (FS) strategy. This approach is motivated by promising results on FS preconditioners for deterministic PDEs obtained by the authors in a series of papers [28, 29, 30, 31]. When used on deterministic PDEs, the FS strategy yields a block structure associated with the physical variables on the physical domain. On the other hand, here the stochastic modes are used in an analogous way as physical variables for the splitting strategy. The FS approach can also be applied directly as a preconditioner for the GMRES algorithm although, as it will be shown, the best computational performances are obtained if FS is used within the framework of geometric multigrid, i.e. on the smoother. To the best of our knowledge, this is the first work to perform a computational analysis of the performances of GMRES preconditioned with geometric multigrid in the framework of SPDEs. Because from