INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 2-3, Pages 255–274

OPTIMAL CONTROL PROBLEM OF AN SIR MODEL WITH RANDOM INPUTS BASED ON A GENERALIZED POLYNOMIAL CHAOS APPROACH

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Abstract. This paper studies the optimal control problem of a susceptibleinfectiousrecovered (SIR) epidemic model with random inputs. We prove the existence and uniqueness of a solution to the SIR random differential equation (RDE) model and investigate the numerical solution to the model by using a generalized polynomial chaos (gPC) approach. We formulate the optimal control problem of the SIR RDE model and consider the gPC Galerkin method to convert the problem into an optimal control problem with high-dimensional ordinary differential equations. Numerical simulations show that to effectively control an epidemic, vaccination should be given at the highest rate in the first few days, and after that, vaccination should be stopped completely. In addition, we observe that the optimal control function and the corresponding states are very robust to the uncertainty of random inputs.

Key words. Optimal control problem, random differential equation, generalized polynomial chaos.

1. Introduction

Mathematical models can help predict the dynamics of disease transmission and evaluate the impact of control measures. The susceptible-infectious-recovered (SIR) model proposed by O. Kermack and A. G. McKendrick in 1927 [18] is a simple deterministic model to describe an epidemic. The model divides the host population into three compartments: susceptible (S), infectious (I), and recovered (R) individuals. Several studies have conducted outstanding surveys of basic compartment models and explored key features of modified models [2, 7, 14].

However, real-world problems often involve uncertainty due to a lack of information or measurement errors in the data, for example. The randomness in probability theory is used to express uncertainty, and stochastic models have been developed to better describe a complex phenomenon. The stochastic differential equation (SDE) and the random differential equation (RDE) are known to be effective tools in epidemiology. The SDE adds white noise to incorporate perturbation, and Ito integration is the key technique for the analysis [12, 19, 25, 26, 27]. Random variables are employed to represent uncertain input, including parameters and initial conditions, and the RDE is introduced as a result. In this research, we consider a random differential equation for the SIR model [3, 31].

There are analytical and numerical approaches to figure out the property of a solution to RDE models. The analytical one finds the probability density function of a solution by using random variable transformation [8, 17, 28, 30]. Among many numerical schemes to solve RDE models, Monte-Carlo simulation is the basic algorithm to characterize the solution. Generalized polynomial chaos (gPC) [9, 32] and the stochastic collocation method [32, 33] are useful choices in particular settings.

Received by the editors August 28, 2021 and, in revised form, November 26, 2021.

2000 *Mathematics Subject Classification.* 34F05, 49K15, 92B05.

Vaccination is one of the crucial interventions for reducing the spread of infectious diseases. In order to minimize disease burdens, it is important to determine the optimal vaccination policies to better allocate limited resources. Previous studies applied optimal control techniques to derive an efficient vaccination strategy for influenza outbreaks under specific circumstances [20, 21, 22, 23, 24, 34]. Lahrouz et al. considered two types of control to reduce the number of infectious individuals: treatment and preventive campaigns to avoid relapses [20]. Modified models were introduced to incorporate seasonal forcing and age-structure, and optimal strategies for vaccination, antiviral treatments, and social distancing were suggested in [21]. Li et al. proposed a model based on complex networks to discuss an effective quarantine scheme [24] and others have investigated the distribution of vaccines under limited resources using a model with group mixing [34]. Many researchers have also applied control theory to develop optimal strategies for other diseases including HIV, tuberculosis, and vector-borne diseases [1, 5, 15, 16].

The goal of this paper is to derive an optimal vaccination strategy using the SIR model with random inputs. To achieve that, the gPC Galerkin method, which can be applied to various numerical techniques and control theory, is employed. In Section 2, we formulate the SIR RDE model with a random transmission rate and initial conditions for a susceptible population. Then, the existence and uniqueness of a solution to the model are analyzed. A brief introduction to gPC in Section 3 is followed by applying the stochastic Galerkin method using an orthogonal polynomial basis to approximate a solution to the RDE model in Section 4. We also compare the gPC Galerkin solution with Monte-Carlo simulation to evaluate the quality of approximation. In Section 5, we explore an optimal control problem that minimizes the number of infected individuals while considering intervention costs. Finally, Section 6 presents the results from numerical simulations with several distributions of random variables, and we conclude with a summary in Section 7.

2. The SIR model with random inputs

In this section, we introduce an SIR model with random coefficients, and we prove the existence and uniqueness of a solution to the model equation. The SIR model consists of three compartments. The susceptible compartment, S, represents individuals who are susceptible to the disease, while the infectious compartment, I, represents infected individuals who can infect susceptible people. The recovered compartment, R, represents individuals who have recovered from the disease or who have been immunized against the disease. The SIR model with random inputs is given by the following random differential equation [3, 30, 31]:

$$\begin{cases} \dot{S}(t) &= -\beta S(t)I(t) - \mu u(t)S(t) \\ \dot{I}(t) &= \beta S(t)I(t) - \gamma I(t) \\ \dot{R}(t) &= \mu u(t)S(t) + \gamma I(t) \end{cases} \tag{1}$$

with initial conditions $S(0) = S_0, I(0) = I_0$ and $R(0) = R_0$.

The positive parameter β denotes the transmission rate of the disease, and μ is vaccination efficacy. Infected individuals leave infectious class I at rate γ. μ and γ are positive constants. The control function $u(t)$ indicates the rate at which susceptible individuals are vaccinated, and its value is assumed to be in the range $[0, 1]$. Usually, the parameters and initial conditions are considered to be constants in the deterministic model. In this paper, it is assumed that infection rate β and the initial value of susceptible compartment S_0 are functions of random variables