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ANALYSIS AND NUMERICAL RESULTS FOR BOUNDARY OPTIMAL CONTROL PROBLEMS APPLIED TO TURBULENT BUOYANT FLOWS

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Abstract. In this work, we introduce the mathematical analysis of the optimal control for the Navier-Stokes system coupled with the energy equation and a k-ω turbulence model. While the optimal control of the Navier-Stokes system has been widely studied in past works, only a few works are based on the analysis of the turbulent flows. Moreover, the optimal control of turbulent buoyant flows are usually not taken into account due to the difficulties arising from the analysis and the numerical implementation of the optimality system. We first prove the existence of the solution of the boundary value problem associated with the studied system. Then we use an optimization method that relies on the Lagrange multiplier formalism to obtain the first-order necessary condition for optimality. We derive the optimality system and we solve it using a gradient descent algorithm that allows uncoupling state, adjoint, and optimality conditions. Some numerical results are then reported to validate the presented theoretical analysis.

Key words. Optimal control, buoyancy, turbulence modeling, k-ω model.

1. Introduction

In recent years, the optimal control of the energy and Navier-Stokes equations has gained attention in a variety of engineering fields. The optimal design of natural or mixed convection systems is crucial in many contexts, ranging from the thermal-hydraulics of nuclear reactors to semiconductor production processes where buoyancy forces control crystal growth.

In past years, considerable progress has been made in the mathematical analysis of the optimal control of Navier-Stokes and energy equations. Several works have been focused on the optimal control of the heat transfer in forced convection flows, where the coupling between the Navier-Stokes and energy equations is a one-way coupling, see for example [1, 2] and citations therein. In the case of natural or mixed convection flows, the mathematical analysis of the optimal control for the Oberbeck-Boussinesq system has been considered in several works focusing on stationary distributed and boundary controls [3, 4, 5, 6]. Distributed controls are very effective, but they are not feasible in many real cases. In the case of distributed controls, a feedback control can be applied over long period of time to obtain steady desired solutions, see for example [7, 8].

In this paper, we consider only boundary steady optimal control problems for turbulent flows in mixed or natural convection. The mathematical analysis and numerical simulations of the optimal control for turbulent flows without considering the temperature dependence have been investigated in past works [9, 10, 11]. An adjoint approach for the optimal control of turbulent buoyancy-driven flows has been proposed in [12], however a mathematical analysis has not been presented.

In this work, we consider the Reynolds averaged Navier-Stokes and energy system. The state is defined by the average velocity, total pressure field $(\mathbf{u}, p)$, the temperature field T and closed with a k-ω turbulence model [13], where k is the

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turbulent kinetic energy and ω its specific dissipation rate. We introduce the symmetric deformation tensor $\mathbf{S}(\mathbf{u})$ and its squared norm $\mathbf{S}^2(\mathbf{u})$ as

$$\mathbf{S}(\mathbf{u}) = \nabla\mathbf{u} + \nabla\mathbf{u}^T\,, \quad \mathbf{S}^2(\mathbf{u}) = \mathbf{S}(\mathbf{u}) : \mathbf{S}(\mathbf{u})\,.$$

The k-ω dynamical production terms S_k and S_ω for turbulence equations are defined by

$$S_k = \nu_t \mathbf{S}(\mathbf{u}) : \nabla\mathbf{u} = \frac{1}{2}\nu_t \mathbf{S}^2(\mathbf{u})\,, \tag{1}$$

$$S_\omega = \frac{\eta\omega}{k}\nu_t \mathbf{S}(\mathbf{u}) : \nabla\mathbf{u} = \frac{1}{2}\eta \mathbf{S}^2(\mathbf{u})\,. \tag{2}$$

We model the flow as incompressible according to the Oberbeck-Boussinesq approximation neglecting fluid density variations risen by the temperature in the advective term. Density temperature dependence cannot be neglected in the buoyancy force and a linear dependence is taken into account through the fluid coefficient of expansion γ around the reference temperature T_0 in the following specific form of the buoyancy force

$$\mathbf{f}_b = \gamma\mathbf{g}(T - T_0)\,,$$

where $\mathbf{g}$ is the gravitational acceleration vector.

The production terms due to buoyancy in k-ω equations are modeled according to [14, 15]. The source terms depending on the interaction between gravity and the turbulent heat flux components are modeled as

$$S_{k,b} = \frac{\gamma\nu_t}{Pr_t}\mathbf{g} \cdot \nabla T\,, \tag{3}$$

$$S_{\omega,b} = \frac{\eta\gamma}{Pr_t}\mathbf{g} \cdot \nabla T\,. \tag{4}$$

The coefficients η, β and β^* are model constants [13].

Under this framework, we consider an open bounded domain Ω with boundary Γ and the following governing state equations

$$(\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p - \nabla \cdot [(\nu + \nu_t)\mathbf{S}(\mathbf{u})] = \mathbf{f} - \gamma\mathbf{g}(T - T_0)\,, \tag{5}$$

$$\nabla \cdot \mathbf{u} = 0\,, \tag{6}$$

$$(\mathbf{u} \cdot \nabla)T = \nabla \cdot [(\alpha + \alpha_t)\nabla T]\,, \tag{7}$$

$$(\mathbf{u} \cdot \nabla)k - \nabla\left[(\nu + \sigma_k\nu_t) \cdot \nabla k\right] = S_k + S_{k,b} - \beta^*\, k\,\omega\,, \tag{8}$$

$$(\mathbf{u} \cdot \nabla)\omega - \nabla\left[(\nu + \sigma_\omega\nu_t) \cdot \nabla \omega\right] = S_\omega + S_{k,b} - \beta\omega^2\,, \tag{9}$$

where ν is the kinematic viscosity, α thermal diffusivity of the fluid, $\nu_t = k/\omega$ is the eddy kinematic viscosity and $\alpha_t = \nu_t/Pr_t$ is the eddy thermal diffusivity, where the turbulent Prandtl number Pr_t is assumed to be constant. The coefficients σ_k and σ_ω are model constants [13]. The system of equations (5)-(9) defines the state variable $(\mathbf{u}, p, T, k, \omega)$ when this is completed with suitable boundary conditions. However, the above system may not have a solution in many physical situations when k and ω become too large or too small. The k and ω equations have the typical pattern of the diffusion-reaction equations and therefore, introducing some assumptions, their solutions can be constrained inside a precise interval limited by the roots of the equation defined only by the right-hand-side non-linear terms. In an infinite medium or when advection and diffusion terms are negligible the equations (8)-(9) reduce to the non-linear right-hand-side terms

$$S_k + S_{k,b} - \beta^* k\omega = 0\,, \tag{10}$$

$$S_\omega + S_{\omega,b} - \beta\omega^2 = 0\,, \tag{11}$$