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NUMERICAL ANALYSIS OF HIGH ORDER TIME STEPPING SCHEMES FOR A PREDATOR-PREY SYSTEM

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This paper is dedicated to Prof. Max Gunzburger on the occasion of his 75th birthday.

Abstract. Finite element discretisations of the modified predator-prey system are examined. In particular, fully-discrete schemes based on the discontinuous Galerkin time stepping approach for the temporal discretisation combined with standard finite elements for the spatial discretisation are considered. Stability estimates are derived for schemes of arbitrary order and error estimates that maintain a symmetric structure are proved.

Key words. Predator-Prey systems, discontinuous Galerkin schemes, stability properties, error estimates.

1. Introduction

The scope of this work is the stability and error analysis of fully-discrete schemes for the predator-prey system. The predator-prey system under consideration, consists of two coupled parabolic pdes, i.e.,

$$\begin{cases} u_t - d_1\Delta u - u(1-|u|) + vh(au) &= 0 \qquad \text{in } (0,T)\times\Omega \\ \dfrac{\partial u}{\partial n} &= 0 \qquad \text{on } (0,T)\times\Gamma \\ u(0,x) &= u_0 \qquad \text{in } \Omega, \end{cases}$$

$$\begin{cases} v_t - d_2\Delta v - bvh(au) + cv &= 0 \qquad \text{in } (0,T)\times\Omega \\ \dfrac{\partial v}{\partial n} &= 0 \qquad \text{on } (0,T)\times\Gamma \\ v(0,x) &= v_0 \qquad \text{in } \Omega. \end{cases}$$

Here, $d_1, d_2 > 0$ denote diffusion constants, with $d_1 \neq d_2$, $b, c, a > 0$ are positive parameters and $\Omega \subset \mathbb{R}^3$ is a bounded domain with suitably smooth boundary Γ. The initial data are denoted by u_0, v_0 respectively. Our analysis covers two of the most commonly used functional responses $h(.)$, the Holling type II and type III functionals, defined by:

$$h(au) = \frac{au}{1+a|u|} \qquad \text{or} \qquad h(au) = \frac{au^2}{1+au^2},$$

respectively, and involves the nonlinear reaction function $u(1-|u|)$. These type of functional responses were proposed in [30, 31]. For an overview of the role of such functional responses in these models we refer the reader to [32]. The above system is often called the "modified predator-prey system". Our goal is to establish stability and error estimates for fully-discrete schemes of arbitrary order. The schemes under consideration are based on a discontinuous Galerkin -in time- approach combined with standard conforming finite elements in space. Such schemes are known to maintain the structural properties of the underlying pde model, in the sense, that it

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possible to prove stability estimates under minimal regularity assumptions. Indeed, given initial data $u_0, v_0 \in L^2(\Omega)$, in Section 3, we prove the following estimate:

$$\|u_h\|_{W(0,T)} + \|v_h\|_{W(0,T)} \leq C \left(\|u_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\Omega)} \right),$$

where u_h, v_h denote the fully-discrete approximations of weak solutions u, v, and $\|.\|_{W(0,T)} := \|.\|_{L^\infty[0,T;L^2(\Omega)]} + \|.\|_{L^2[0,T;H^1(\Omega)]}$ denotes the natural energy norm associated to the discontinuous Galerkin approximation in time. The key difficulty involves the derivation of estimates for higher order schemes at the $\|.\|_{L^\infty[0,T;L^2(\Omega)]}$ norm in presence of the nonlinear coupling. We note that the above stability estimate under minimal regularity assumptions, is the key step in order to develop a-priori error estimates. In addition, for $u, v \in W(0,T) \cap L^\infty[0,T;L^\infty(\Omega)]$ we establish the fully-discrete analog of the classical Céa Lemma which in this context, is an estimate of the form,

$$\|u - u_h\|_{W(0,T)} + \|v - v_h\|_{W(0,T)} \leq C \Big(\|u - P_h^{loc} u\|_{W(0,T)} + \|v - P_h^{loc}\|_{W(0,T)} \Big).$$

Here, P_h^{loc} denotes the standard projection associated to discontinuous Galerkin schemes that exhibits best approximation properties in terms of the available regularity of the solution.

We emphasise that this estimate is also derived under minimal regularity assumptions on data and it is applicable when high order schemes are employed. Such estimate demonstrates that the error of the fully-discrete scheme will convergence at the maximal rate that the chosen approximation spaces and the regularity of the solution will allow. The estimate is valid for a suitable choice of the temporal discretisation parameter τ in terms of the parameters a, b, c, d_1, d_2, but it can be chosen independent of the size of the spatial discretisation parameter h. Our work uses ideas and techniques of [9, 8], developed for proving estimates at arbitrary time points, combined with a suitable "boot-strap" argument that decouples the two involved pdes without imposing additional regularity and / or very stringent conditions between the discretisation parameters and the physical parameters of our system. In addition the estimate is derived without making assumptions regarding point-wise space-time stability on u_h, v_h. To our best knowledge these estimates are new.

Various issues related to numerical analysis and computational efficiency of discretisation schemes for systems of reaction-diffusion pdes that resemble the predator-prey system have been considered before (see e.g. [5, 6, 7, 14, 23, 24, 25, 27, 28, 33, 34, 35, 36, 40]). In particular, we point out [27] where a-priori estimates are established for the fully-discrete approximation of the predator-prey system using semi-implicit Euler scheme in time combined with conforming finite elements in space and [14] where the analysis of first-order in time implicit-symplectic method is considered.

Stability analysis and a-priori error estimates involving the Brusselator nonlinear coupling structure are presented in the work of [7]. A finite volume scheme for the Brusselator model with cross diffusion is considered in [35], while an alternative direction (ADI) extrapolated Crank-Nicolson orthogonal collocation algorithm is analysed in [23]. Both papers include various informative computational results and are applicable in other nonlinear reaction diffusion systems. In [40], implicit-explicit schemes for various reaction diffusion systems arising in pattern formation are considered, while analytical and computational aspects of moving grid time-stepping schemes are studied in [36]. In [24], optimal error bounds of a fully-discrete scheme based on the implicit-explicit Euler method combined with