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Sharp Convergence to Steady States of Allen-Cahn

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Abstract. In our recent work we found a surprising breakdown of symmetry conservation: using standard numerical discretization with very high precision the computed numerical solutions corresponding to very nice initial data may converge to completely incorrect steady states due to the gradual accumulation of machine round-off error. We solved this issue by introducing a new Fourier filter technique for solutions with certain band gap properties. To further investigate the attracting basin of steady states we classify in this work all possible bounded nontrivial steady states for the Allen-Cahn equation. We characterize sharp dependence of nontrivial steady states on the diffusion coefficient and prove strict monotonicity of the associated energy. In particular, we establish

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a certain self-replicating property amongst the hierarchy of steady states and give a full classification of their energies and profiles. We develop a new modulation theory and prove sharp convergence to the steady state with explicit rates and profiles.

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Key words: Allen-Cahn equation, steady state, ground state solution, asymptotic behavior.

1 Introduction

In this paper, we consider the following one-dimensional Allen-Cahn equation posed on the periodic torus $\mathbb{T}=[-\pi,\pi]$:

$$\begin{cases} \partial_t u = \kappa^2 \partial_{xx} u - f(u), \\ u\big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $\kappa>0$ measures the strength of diffusion, $f(u)=u^3-u=F'(u)$, and $F(u)=\frac{(u^2-1)^2}{4}$ is the usual double-well potential. The function $u:\mathbb{T}\to\mathbb{R}$ represents the concentration difference of phases in an alloy and typically has values in the physical range $[-1,1]$.

In our recent work [13], we find a very surprising breakdown of parity in typical high-precision computation of (1.1) with very smooth initial data. For example take $\kappa=1$ and consider Eq. (1.1) with the initial data $u_0(x)$ being an odd function of x such as $u_0(x)=\sin x$. By simple PDE arguments the smooth solution should preserve the odd symmetry for all time. However numerical discretized solutions turn out to fail to conserve this parity and converge quickly to the spurious states $u=\pm 1$ in not very long time simulations. This striking contradiction is a manifestation of the gradual accumulation of non-negligible machine round off errors over time. To resolve this issue, we introduced a new Fourier filter method which works successfully for a class of initial data with certain symmetry and band-gap properties. By eliminating the unwanted projections into the unstable directions at each iteration, we rigorously show (see [12, 13]) that the filtered solution will converge to the true steady state in long time simulations.

A natural next task is to understand the situation for general solutions without symmetries or band-gap properties. The pivotal step is to categorize the steady states of the elliptic Allen-Cahn equations and analyze in detail their spectral