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BOUNDARY INTEGRAL EQUATIONS FOR ISOTROPIC LINEAR ELASTICITY*

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Abstract

This articles first investigates boundary integral operators for the three-dimensional isotropic linear elasticity of a biphasic model with piecewise constant Lamé coefficients in the form of a bounded domain of arbitrary shape surrounded by a background material. In the simple case of a spherical inclusion, the vector spherical harmonics consist of eigenfunctions of the single and double layer boundary operators and we provide their spectra. Further, in the case of many spherical inclusions with isotropic materials, each with its own set of Lamé parameters, we propose an integral equation and a subsequent Galerkin discretization using the vector spherical harmonics and apply the discretization to several numerical test cases.

Mathematics subject classification: 65R20, 65N38, 74B05.
Key words: Isotropic elasticity, Boundary integral equation, Spherical inclusions, Vector spherical harmonics, Layer potentials.

1. Introduction

We consider three-dimensional boundary value or interface problems of the isotropic elasticity equation related to the following operator:

$$\mathbf{L}u := -\operatorname{div}\Big(2\mu e(u) + \lambda \operatorname{Tr}\big(e(u)\big)\operatorname{Id}\Big), \tag{1.1}$$

where the strain tensor reads $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$. It is obvious to see that the operator $\mathbf{L}$ is self-adjoint on $L^2(\mathbb{R}^3)^3$.

In the definition of the operator (1.1), $\mu, \lambda \in \mathbb{R}, \mu > 0, 2\mu + 3\lambda > 0$ are the so-called (constant) Lamé parameters. The parameter μ denotes the shear modulus which describes the tendency of the object to deform at a constant volume when being imposed with opposing forces. The other Lamé parameter λ has no physical meanings but is introduced to simplify the definition of the operator (1.1). Indeed, it is related to the bulk modulus K through the relation

$$\lambda = K - \frac{2}{3}\mu,$$

where the bulk modulus K represents the object's tendency to deform in all directions when acted on by opposing force from all directions. We refer to [12] for more detailed descriptions

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of the Lamé parameters. It is sometimes useful to introduce Poisson's ratio ν which is defined by

$$\nu = \frac{\lambda}{2(\mu + \lambda)}, \tag{1.2}$$

and whose admissible range is $(-1, 1/2)$. The material is extremely compressible in the limit $\nu \to -1$ while extremely incompressible in the other limit $\nu \to 1/2$ [18].

A model of linear elasticity with appropriate boundary conditions can be approximated by the classic finite element method, see for example [15,19] just to name a few contributions from an abundant body of literature, for the general case with non-homogeneous source term. On the other hand, displacement fields u being homogeneous solutions, i.e., $\mathbf{L}u = 0$ within a given domain, can also be represented by isotropic elastic potentials [3,13] and elasticity in piecewise constant isotropic media can then be treated as integral equations for specified interface conditions. At the origin of the integral formulation lies the definitions of layer potentials and their corresponding integral operators [3,20] based on the Green's function [1] in the context of the isotropic linear elasticity.

In particular, on a unit sphere, one can introduce the vector spherical harmonics forming an orthonormal basis of $[L^2(\mathbb{S}^2)]^3$ and which are eigenfunctions of the corresponding double and single layer boundary operators based on the Green's function [1] of isotropic linear elasticity. The vector spherical harmonics were introduced in [9,10] as an extension of the scalar spherical harmonics [16,23] to the vectorial case. They were further used in the discretization of different physical models such as the Navier-Stokes equations [8] or Maxwell's equations [2,6]. However, they are not widely used and only sparely reported in literature, in particular in the context of isotropic elasticity. We demonstrate in this article that the corresponding integral operators have interesting spectral properties which can be made explicit by employing the vector spherical harmonics.

Our main motivation for this work is the derivation of an integral equation to model elastic materials represented by piecewise constant Lamé constants with spherical inclusions following similar principles that were presented in [4,5,14] in the case of scalar diffusion. The particular choice of the vector spherical harmonics as basis functions for a Galerkin discretization thereof leads then to an efficient and stable numerical scheme by exploiting the spectral properties of the involved integral operators. A similar physical model was introduced in [22] with an algebraic formula of the approximate solution. However, with the spectral properties of the layer potentials and integral operators at hand, our approach first introduces an integral formulation for the exact solution and thus a rigorous mathematical framework. In a second step, we then propose the Galerkin discretization. The mathematical framework lays out the basis to derive a rigorous error analysis which we plan in the future.

We summarize the main contributions and organization of this work as follows:

- In Sections 2 and 3, we give an introduction and overview of the layer potentials and corresponding boundary integral operators of the isotropic linear elasticity operator (1.1) on an arbitrary bounded domain with Lipschitz boundary which are sparely reported in the literature.

- Analytical properties of layer potentials and boundary integral operators are presented and proven in Section 3.4.

- On the unit sphere, we introduce the vector spherical harmonics in Section 4 and prove spectral properties of the boundary operators and layer potentials of this particular basis.