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# A SECOND-ORDER EMBEDDED LOW-REGULARITY INTEGRATOR FOR THE QUADRATIC NONLINEAR SCHRÖDINGER EQUATION ON TORUS

FANGYAN YAO

**Abstract.** A new embedded low-regularity integrator is proposed for the quadratic nonlinear Schrödinger equation on the one-dimensional torus. Second-order convergence in $H^\gamma$ is proved for solutions in $C([0,T];H^\gamma)$ with $\gamma > \frac{3}{2}$, i.e., no additional regularity in the solution is required. The proposed method is fully explicit and can be computed by the fast Fourier transform with $\mathcal{O}(N\log N)$ operations at every time level, where $N$ denotes the degrees of freedom in the spatial discretization. The method extends the first-order convergent low-regularity integrator in [14] to second-order time discretization in the case $\gamma > \frac{3}{2}$ without requiring additional regularity of the solution. Numerical experiments are presented to support the theoretical analysis by illustrating the convergence of the proposed method.

**Key words.** Quadratic nonlinear Schrödinger equation, low-regularity integrator, second-order convergence, fast Fourier transform.

## 1. Introduction

This paper is concerned with the development of low-regularity integrators for the quadratic nonlinear Schrödinger (NLS) equation on the one-dimensional torus, i.e.,

$$\begin{cases} i\partial_t u(t,x) + \partial_{xx} u(t,x) = \mu u^2(t,x), & t>0 \text{ and } x\in\mathbb{T}=[0,2\pi], \\ u(0,x) = u^0(x). \end{cases} \tag{1}$$

where $u:\mathbb{R}^+\times\mathbb{T}\to\mathbb{C}$ is a complex-valued unknown function with initial value $u^0\in H^\gamma(\mathbb{T})$, $\gamma\geq 0$, and $\mu\in\mathbb{R}$ is a given constant. The well-posedness of the equation has been proved in [1].

Time discretization of the nonlinear Schrödinger equation has been considered in many papers with different methods. In general, classical time discretizations require the solution to be in $C([0,T];H^{\gamma+2})$ and $C([0,T];H^{\gamma+4})$ in order to have first- and second-order convergence in $H^\gamma$, respectively, i.e., two additional derivatives in the solution are required for every order of convergence. The convergence of time discretizations under these (or stronger) regularity conditions has been proved for the finite difference methods [17], operator splitting [2, 5, 10], and exponential integrators [4].

In practical computations, the initial data may be polluted by nonsmooth noises from the measurements. Accordingly, the development of low-regularity integrators which can reduce the regularity requirement of the solution and has attracted much attention from numerical analysts. Ostermann and Schratz [14] proposed a new exponential-type integrator for the cubic NLS equation in the $d$-dimensional space, and proved its first-order convergence in $H^\gamma$ for solutions in $C([0,T];H^{\gamma+1})$, with $\gamma > \frac{d}{2}$. In one dimension, Wu and Yao [18] proposed a new time discretization which has first-order convergence in $H^\gamma$ for solutions

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in $C([0,T];H^\gamma)$ with $\gamma > \frac{3}{2}$, without requiring any additional regularity in the solution. These articles are all concerned with first-order convergent low-regularity integrators.

Knöller, Ostermann and Schratz [7] proposed a second-order low-regularity integrator which requires two and three additional derivatives in the solution in one- and higher-dimensional spaces, respectively. In two- and higher-dimensional spaces, the regularity requirement was relaxed to two additional derivatives by Bruned and Schratz [3] and Ostermann, Wu and Yao [15] with different methods.

For convergence in $L^2$, Ostermann, Rousset and Schratz [12, 13] proved certain fractional-order convergence of some filtered methods for solutions in $C([0,T];H^\gamma)$ with $\gamma \in (0,1]$. Li and Wu [8] constructed a fully discrete low-regularity integrator with first-order convergence in both time and space for solutions in $C([0,T];H^1)$. Ostermann and Yao [16] proposed a different fully discrete method with an error estimate of $\mathcal{O}(\tau^{\frac{3}{2}\gamma-\frac{1}{2}-\varepsilon} + N^{-\gamma})$ for solutions in $C([0,T];H^\gamma)$ with $\gamma \in (\frac{1}{2},1]$.

More recently, Wu and Zhao [19, 20] introduced an embedded low-regularity integrator for the Korteweg-de Vries (KdV) equation with first- and second-order convergence in $H^\gamma(\mathbb{T})$ for solutions in $C([0,T];H^{\gamma+1})$ and $C([0,T];H^{\gamma+3})$, respectively. By using new harmonic analysis techniques, Li, Wu and Yao [9] proposed a method for the KdV equation with $\frac{1}{2}$-order convergence in $H^\gamma$ for solutions in $C([0,T];H^\gamma)$ with $\gamma > \frac{3}{2}$, without requiring any additional derivatives in the solution. For the modified KdV equation, Ning, Wu and Zhao [11] proposed a new embedded low-regularity integrator and proved first-order convergence by requiring the boundedness of one additional spatial derivative of the solution.

For the quadratic nonlinear Schrödinger equation on the one-dimensional torus, Ostermann and Schratz [14] proposed a low-regularity integrator with first-order convergence in $H^\gamma$ for solutions in $C([0,T];H^\gamma)$, $\gamma > \frac{1}{2}$. In the present paper, we propose a new embedded low-regularity integrator with second-order convergence in $H^\gamma$ for solutions in $C([0,T];H^\gamma)$, $\gamma > \frac{3}{2}$. The construction of the method extends the low-regularity integrators in [19] and [15], which were originally proposed for the KdV equation and cubic nonlinear Schrödinger equation, respectively. The proof of convergence for the proposed method is based on harmonic analysis techniques.

The rest of this paper is organized as follows. The notations and the main result are presented in Section 2. The construction of the low-regularity integrator and the technical lemmas to be used in the convergence analysis are presented in Section 3. The proof of the main theorem is presented in Section 4. Numerical experiments are reported in Section 5. Some concluding remarks are presented in Section 6.

## 2. Notations and main results

**2.1. Some notations.** We denote by $\langle\cdot,\cdot\rangle$ the inner product of $L^2 = L^2(\mathbb{T})$, i.e.,

$$\langle f,g\rangle = \int_{\mathbb{T}} f(x)\overline{g(x)}\,dx, \qquad f,\, g \in L^2.$$

The Fourier transform $(\hat{f}_k)_{k\in\mathbb{Z}}$ of a function $f\colon \mathbb{T} \to \mathbb{C}$ is defined by

$$\hat{f}_k = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx} f(x)\,dx.$$

The inverse Fourier transform formula is given by

$$f(x) = \sum_{k\in\mathbb{Z}} \hat{f}_k e^{ikx}.$$