INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 19, Number 6, Pages 777–792

ASYMPTOTIC AND EXACT SELF-SIMILAR EVOLUTION OF A GROWING DENDRITE

AMLAN K. BARUA, SHUWANG LI, XIAOFAN LI, AND PERRY LEO

Abstract. In this paper, we investigate numerically the *long-time* dynamics of a two-dimensional dendritic precipitate. We focus our study on the self-similar scaling behavior of the primary dendritic arm with profile $x \sim t^{\alpha_1}$ and $y \sim t^{\alpha_2}$, and explore the dependence of parameters α_1 and α_2 on applied driving forces of the system (e.g. applied far-field flux or strain). We consider two dendrite forming mechanisms: the dendritic growth driven by (i) an anisotropic surface tension and (ii) an applied strain at the far-field of the elastic matrix. We perform simulations using a spectrally accurate boundary integral method, together with a rescaling scheme to speed up the intrinsically slow evolution of the precipitate. The method enables us to accurately compute the dynamics far longer times than could previously be accomplished. Comparing with the original work on the scaling behavior $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$ [Phys. Rev. Lett. 71(21) (1993) 3461–3464], where a constant flux was used in a diffusion only problem, we found at long times this scaling still serves a good estimation of the dynamics though it deviates from the asymptotic predictions due to slow retreats of the dendrite tip at later times. In particular, we find numerically that the tip grows self-similarly with $\alpha_1 = 1/3$ and $\alpha_2 = 1/3$ if the driving flux $J \sim 1/R(t)$ where $R(t)$ is the equivalent size of the evolving precipitate. In the diffusive growth of precipitates in an elastic media, we examine the tip of the precipitate under elastic stress, under both isotropic and anisotropic surface tension, and find that the tip also follows a scaling law.

Key words. Moving boundary problems, self-similar, dendrite growth, boundary integral equations.

1. Introduction

The evolution of precipitates during a solid-solid phase transformation is a classical example for studying interface dynamics or systems driven out of equilibrium. A well-known feature observed during the phase transformation is the formation of various dendritic microstructures, depending on the physical conditions (e.g. the composition of the phases, the interfacial crystallographic properties and the applied far-field flux). Usually a dendrite includes the primary arm (tip region) and accompanied side-branches. One key aspect of studying the precipitate morphology is to understand the evolution of tip profile, as its dynamics determines the resulting morphology of the dendrite.

An early theory trying to describe the dendritic tip is due to Ivantsov [4] who assumed that the region near the tip of a dendrite is a branch-less paraboloid growing with a constant velocity. These assumptions allow him to solve the steady state heat transport equation and establish an analytical relation between the Stefan number and the Peclet number, the two dimensionless quantities important for the process. A large number of work was built upon this original formulation. For example, capillary effects were coupled to heat transfer problem through Gibbs-Thomson condition due to the work of Nash and Glicksmann [21, 22]. In a more recent paper [19], Lacombe et. al. showed that paraboloid shape assumption was not valid if one moves slightly away from the tip and to a region where the side

Received by the editors February 27, 2022 and, in revised form, June 2, 2022; accepted June 19, 2022.

2000 *Mathematics Subject Classification.* 52B10, 65D18, 68U05, 68U07.

branches emanate. They showed that a much better match with experiments occurs if a fourth-order correction, in terms of radius of curvature of the tip, is applied to the predictions of Ivantsov. Ivantsov's original formulation did not take into account the effects of side-branching, however considerable research has been done in this area where the main question is to understand the frequency and and amplitude of the secondary branches. These questions are tackled roughly through two approaches. A few authors suggest that the deterministic oscillation at the tip is responsible for side-branching [6, 15, 23]. Others explain the mechanism via stochastic approach and consider selective, thermal fluctuation induced noise to be responsible for side-branching process [2, 5, 8, 10, 14, 16].

Li and Beckerman [1] studied the scaling behavior of both the tip and side branches with different geometric parameters by performing micro-gravity experiments using pure succonitrile crystals. Their experimental results were in good agreement with the theoretical predictions of [2]. A more recent work on dendrite morphology using boundary integral methods can be found in [20, 27] where deterministic side-branching mechanism for 2D and 3D growths were considered.

A seemingly different problem, the Hele-Shaw flow, should also be mentioned in this context. Although the origins of the Hele-Shaw problem lie in creeping flow between two closely placed parallel plates, the flow is governed by similar equations. Interesting results that emerge in a Hele-Shaw cell can be found in [9] where the author investigated formation of different patterns of a growing bubble both in isotropic and anisotropic surface tension. Numerically, the boundary integral method has been the most successful approach in Hele-Shaw flow where long dynamics, both in growing and shrinking interfaces, has been tracked with highly accurate computation in references such as [26,28,29]. Almgren et al. [30] used ideas from selection theory to argue that $\kappa^2 V$, where κ is the tip curvature and V is the tip velocity, should be time independent for a Hele-Shaw bubble. They assumed that a precipitate in its later phase of growth assumes a cross like shape where one can ignore the lateral width of the arms of the precipitate in comparison to the arm-length. From these considerations they derived a scaling law $(x, y) \rightarrow (x/t^{\alpha_1}, y/t^{\alpha_2})$ for the growing tip, where t is the elapsed time and α_1 and α_2 are parameters. The sum $\alpha_1 + \alpha_2$ depends on the flux J of the incoming material. For constant flux $J = c$ they found these constants to be $\alpha_1 = 0.60$ and $\alpha_2 = 0.40$. Their simulations with moderate and high anisotropy using boundary integral formulation indeed showed the scaling to be true, however they investigated the precipitate growth for a very short time duration. The scaling law was verified experimentally by Ignes-Mullol et al. [17] and very good agreement between the simulation and experiment was observed. In fact their experimentally verified exponent α_1 turned out to be 0.64 which is just slightly off from the theoretical predictions of [30].

In this paper, we expanded the original work of Almgren et al. [30] by answering several interesting questions that emerge naturally from their work. These are: (i) whether the scaling law is valid at long times; (ii) what could be a scaling law for the tip of a precipitate growing under time dependent flux and finally (iii) what happens when precipitate grows in presence elastic fields, i.e. whether the tip still exhibits scaling behavior? Our numerical results suggest that at long times, the Almgren's scaling law still provides a good estimate of the tip-profile although it deviates from the asymptotic predictions due to slow retreat of the dendrite tip at later times. In particular, we find that the tip grows self-similarly with $\alpha_1 = 1/3$ and $\alpha_2 = 1/3$ if the driving flux $J \sim 1/R$ where R is the equivalent radius of the precipitate size. In the diffusive growth of precipitates, we observe the tip of