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# Well-Balanced Central Scheme for the System of MHD Equations with Gravitational Source Term

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**Abstract.** A well-balanced second order finite volume central scheme for the magnetohydrodynamic (MHD) equations with gravitational source term is developed in this paper. The scheme is an unstaggered central scheme that evolves the numerical solution on a single grid and avoids solving Riemann problems at the cell interfaces using ghost staggered cells. A subtraction technique is used on the conservative variables with the support of a known steady state in order to manifest the well-balanced property of the scheme. The divergence-free constraint of the magnetic field is satisfied after applying the constrained transport method (CTM) for unstaggered central schemes at the end of each time-step by correcting the components of the magnetic field. The robustness of the proposed scheme is verified on a list of numerical test cases from the literature.

**AMS subject classifications**: 65M08, 76M12, 65M22

**Key words**: MHD equations, unstaggered central schemes, well-balanced schemes, steady states, divergence-free constraint, constrained transport method.

## 1 Introduction

Ideal Magnetohydrodynamics (MHD) equations model problems in physics and astrophysics. The MHD system is a combination of the Navier-Stokes equations of fluid dynamics and the Maxwell equations of electromagnetism. A gravitational source term is added to the ideal MHD equations in two space dimensions in order to model more complicated problems arising in astrophysics and solar physics such as modeling wave propagation in idealized stellar atmospheres [3, 16]. From electromagnetic theory, the

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magnetic field $\mathbf{B}$ must be solenoidal i.e. $\nabla\cdot\mathbf{B}=0$ at all times. The divergence-free constraint on the magnetic field reflects the fact that magnetic monopoles have not been observed in nature. The induction equation for updating the magnetic field imposes the divergence on the magnetic field. Hence, a numerical scheme for the MHD equations should maintain the divergence-free property of the discrete magnetic field at each time-step. Numerical schemes usually fail to satisfy the divergence-free constraint and numerical instabilities and unphysical oscillations may be observed [17]. Several methods were developed to overcome this issue. The projection method, in which the magnetic field is projected into a zero divergence field by solving an elliptic equation at each time step [5].

Another procedure is the Godunov-Powell procedure [7,13,15], where the Godunov-Powell form of the system of the MHD equations is discretized instead of the original system. The Godunov-Powell system has the divergence of the magnetic field as a part of the source term. Hence, divergence errors are transported out of the domain with the flow.

A third approach is the constrained transport method (CTM) [4,6,14]. The CTM was modified from its original form to the case of staggered central schemes [1]. It was later extended to the case of unstaggered central schemes [19]. Hence, a numerical scheme for the MHD equations should maintain the divergence-free property of the discrete magnetic field at each time-step. A finite volume second-order accurate unstaggered central scheme is used to model the MHD equations with a gravitational source term. Finite volume central schemes were first introduced in 1990 by Nessyahu and Tadmor (NT) [11]. The NT scheme is based on evolving piecewise linear numerical solution on two staggered grids. The most significant property of central schemes is that they avoid solving Riemann problems arising at the cell interfaces. Our scheme is unstaggered central (UC) type scheme that was first developed in [9,18]. These schemes allow the evolution of the numerical solution on a single grid instead of using two different grids. UC schemes were first developed for hyperbolic systems of conservation laws and then extended to hyperbolic systems of balance laws [19–22]. The UC schemes introduced the possibility of avoiding solving Riemann problems and switching between two grids. The approach is achieved by the help of ghost staggered cells used implicitly to avoid Riemann problems at the cell interfaces.

In the presence of a gravitational source term on the right hand side of the MHD system, one has to consider a well-balanced technique that provides the numerical scheme with the ability to preserve hydrostatic equilibrium. In this paper we extend the reconstruction technique on the conservative variables, previously developed in [2,10] for the system of Euler equations, for the system of MHD equations. The well-balancing property and the divergence-free property of the scheme result from the combination of the subtraction method with the CTM, which has not been done in [2]. The idea is to evolve the error function between the vector of conserved variables and a given steady state, instead of evolving the vector of conserved variables. This error function is defined as $\Delta\mathbf{U}=\mathbf{U}-\tilde{\mathbf{U}}$, where $\tilde{\mathbf{U}}$ is a given steady state. Knowing the steady state (analytically or numerically) is a key ingredient for the implementation of the proposed scheme.