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Efficient Hermite Spectral-Galerkin Methods for Nonlocal Diffusion Equations in Unbounded Domains

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Abstract. In this paper, we develop an efficient Hermite spectral-Galerkin method for nonlocal diffusion equations in unbounded domains. We show that the use of the Hermite basis can de-convolute the troublesome convolutional operations involved in the nonlocal Laplacian. As a result, the "stiffness" matrix can be fast computed and assembled via the four-point stable recursive algorithm with $\mathcal{O}(N^2)$ arithmetic operations. Moreover, the singular factor in a typical kernel function can be fully absorbed by the basis. With the aid of Fourier analysis, we can prove the convergence of the scheme. We demonstrate that the recursive computation of the entries of the stiffness matrix can be extended to the two-dimensional nonlocal Laplacian using the isotropic Hermite functions as basis functions. We provide ample numerical results to illustrate the accuracy and efficiency of the proposed algorithms.

AMS subject classifications: 65N35, 65N25, 33C45, 65M70

Key words: Nonlocal diffusion equation, spectral-Galerkin, Hermite functions, correlation/convolution, recurrence algorithm.

1. Introduction

Mathematical models involving nonlocal operators such as fractional integrals/derivatives, fractional Laplacian and nonlocal Laplacian, have proven to be of great value and superior to conventional models in modeling many abnormal physical phenomena and engineering processes [6, 8]. Although these nonlocal operators are defined in

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different senses, they have interwoven connections, and share some common numerical difficulties, e.g., the global dependence and the involvement of singular kernels. In general, a nonlocal operator takes the form

$$\mathcal{L}_{\mathcal{K}}u(\boldsymbol{x}) = \text{P.V.}\int_{\mathbb{R}^d} \big(u(\boldsymbol{x}) - u(\boldsymbol{y})\big)\mathcal{K}\big(|\boldsymbol{x}-\boldsymbol{y}|\big)\mathrm{d}\boldsymbol{y}, \quad \boldsymbol{x}\in\mathbb{R}^d, \tag{1.1}$$

where the kernel $\mathcal{K}:\mathbb{R}^d \to (0,\infty)$ satisfies

$$\begin{aligned} &\gamma\mathcal{K}\in L^1(\mathbb{R}^d) \quad \text{with} \quad \gamma(z)=\min(1,|z|^2), \\ &\exists\,\theta,\, s\in(0,1) \quad \text{such that} \quad \mathcal{K}(\boldsymbol{z})\ge \theta|\boldsymbol{z}|^{-(d+2s)}, \quad \boldsymbol{z}\in\mathbb{R}^d\setminus\{0\}. \end{aligned} \tag{1.2}$$

For example, for the hypersingular integral fractional Laplacian $(-\Delta)^s$, we have

$$\mathcal{K}(\eta) = \frac{2^{2s}s\Gamma(s+d/2)}{\pi^{d/2}\Gamma(1-s)}\eta^{-(d+2s)}. \tag{1.3}$$

The nonlocal Laplacian operator generally reads

$$\mathcal{L}_\delta[u](\boldsymbol{x}) = \int_{\mathbb{B}_\delta^d} \big(u(\boldsymbol{x}+\boldsymbol{s}) - u(\boldsymbol{x})\big)\gamma_\delta\big(|\boldsymbol{s}|\big)\mathrm{d}\boldsymbol{s}, \quad \boldsymbol{x}\in\mathbb{R}^d. \tag{1.4}$$

Here, $\gamma_\delta(z)$ is a nonnegative compactly supported kernel whose support is contained in $[0,\delta]$ and $\mathbb{B}_\delta^d$ is a d-dimensional ball of radius δ.

The properties and applications of the nonlocal operator have been extensively investigated. We refer to [8–10] for a comprehensive exposition of the nonlocal calculus and nonlocal diffusion problems with volume constraints. The δ-compatible studies (i.e., the limit case when $\delta\to 0$) at both continuous and discrete levels were conducted in [29, 30] and some other literature. In fact, when $\delta\to\infty$, it demonstrates that the nonlocal operator can become fractional Laplacian operator in [7]. Many methods and schemes have been exploited to approximate the nonlocal operator, such as domain decomposition method [2], asymptotically compatible schemes [31], discontinuous Galerkin methods [32], and Fourier spectral method [12] among others. Not restricted to these methods, more attempts have been made in studying the solutions of various equations comprising the nonlocal operator, including nonlocal equations with Dirichlet boundaries [15], nonlocal wave equations [4] and nonlocal Allen-Cahn type equations [5].

While most existing works are on the nonlocal problems in bounded domains (and many are for one spatial dimension), there has been much less concern about nonlocal models in the unbounded domain, where such nonlocal operators are naturally set without complications from the boundary. It is noteworthy that a nonlocal diffusion equation on the real line is considered in [36], where the infinite interval was reduced to a finite one by an artificial layer based on the z-transform. The study of the nonlocal analogue of artificial boundary conditions/layers is also a subject of interest [11, 13, 35], but the rigorous error analysis and many other aspects are still worthy of deeper investigation.