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An Energy Stable Filtered Backward Euler Scheme for the MBE Equation with Slope Selection

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Abstract. As a promising strategy to adjust the order in the variable-order BDF algorithm, a time filtered backward Euler scheme is investigated for the molecular beam epitaxial equation with slope selection. The temporal second-order convergence in the L^2 norm is established under a convergence-solvability-stability (CSS)-consistent time-step constraint. The CSS-consistent condition means that the maximum step-size limit required for convergence is of the same order to that for solvability and stability (in certain norms) as the small interface parameter $\varepsilon \to 0^+$. Similar to the backward Euler scheme, the time filtered backward Euler scheme preserves some physical properties of the original problem at the discrete levels, including the volume conservation, the energy dissipation law and L^2 norm boundedness. Numerical tests are included to support the theoretical results.

AMS subject classifications: 65M06, 65M12

Key words: MBE model, time filter, energy dissipation law, error estimate.

1. Introduction

Filtering algorithm is a kind of numerical post-processing algorithm based on the original calculation code of complex system. It is widely used in computational fluid industry applications [1,4] and numerical weather forecasting [23,24] to improve numerical simulation, such as eliminating high-frequency oscillations to improve stability, reducing dispersion error to improve computational accuracy, etc. Recently, the effect of adding a simple time filter to backward Euler method is considered in [6] for the initial value problem

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$$y'(t) = f\big(t, y(t)\big) \quad \text{for} \quad t > 0, \quad y(0) = y_0.$$

Consider the uniform time level $t_k = k\tau$ for $0 \le k \le N$ with the time-step size $\tau := T/N$. Given a grid sequence $\{v^n\}_{n=0}^N$, denote

$$\partial_\tau v^n := \frac{1}{\tau}(v^n - v^{n-1}), \quad \partial_\tau^2 v^n := \frac{1}{\tau}\big(\partial_\tau v^n - \partial_\tau v^{n-1}\big).$$

The time filtered backward Euler (FiBE) method in [6] approximates this problem by the backward Euler method and uses a simple time filter to update the solution

$$\begin{aligned} &\text{Step1}: \quad \frac{1}{\tau}\big(y_*^n - y^{n-1}\big) = f(t_n, y_*^n) && \text{for} \quad n \ge 2, \\ &\text{Step2}: \quad y^n = y_*^n - \frac{\nu}{2}\big(y_*^n - 2y^{n-1} + y^{n-2}\big) && \text{for} \quad n \ge 2, \end{aligned}$$

where y_*^n and y^n denote unfiltered and filtered values, and ν is an algorithm parameter to be determined. The FiBE method improves the accuracy of the fully implicit method to second-order by a well-calibrated post-filter with $\nu = 2/3$. In the absence of a better approach, time filter method solves the problem of the accuracy improvement in a complex, possibly legacy code, and the approach is modular and requires the addition of only one line of additional code. Error estimation and variable time step are straightforward and the individual effect of each step is conceptually clear. Recently, this method was extended to the Navier-Stokes equations in [4]. DeCaria *et al.* [3] presented several new embedded families of high accuracy methods with low cognitive complexity and excellent stability properties.

As seen, the above time filtered approach is a promising strategy to adjust the order in variable-order and variable-step BDF algorithms for gradient flow models, see our recent analysis [7, 9–15] on the variable-step BDF2 and high-order BDF methods. As pointed out in [6], the combination of backward Euler plus a curvature reducing time filter gives another option for long-time numerical simulations although the BDF2 method is satisfactory in many applications. To this aim, the stability and convergence of the FiBE method is investigated for the molecular beam epitaxy (MBE) model with slope selection [18], which can be viewed as an L^2 gradient flow of the Ehrlich-Schwoebel energy functional

$$E[\Phi] = \int_\Omega \left[\frac{\varepsilon^2}{2}|\Delta\Phi|^2 + F(\nabla\Phi)\right] \mathrm{d}\boldsymbol{x}, \tag{1.1}$$

where

$$F(\boldsymbol{v}) = \frac{1}{4}\big(|\boldsymbol{v}|^2 - 1\big)^2$$

is a nonlinear energy density function. In recent years, many of time stepping methods, including the stabilized semi-implicit scheme [25], the Crank-Nicolson scheme [19], the convex splitting scheme [20], and operator splitting schemes [8, 16, 17, 27] have been constructed and analyzed for the MBE growth model and related nonlinear models, also see [2, 5, 7, 12, 21, 22] and references therein.