INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 2, Pages 199–228

doi: 10.4208/ijnam2023-1009

WEAK GALERKIN FINITE ELEMENT METHODS FOR PARABOLIC PROBLEMS WITH L^2 INITIAL DATA

NARESH KUMAR AND BHUPEN DEKA

Abstract. We analyze the weak Galerkin finite element methods for second-order linear parabolic problems with L^2 initial data, both in a spatially semidiscrete case and in a fully discrete case based on the backward Euler method. We have established optimal L^2 error estimates of order $\mathrm{O}(h^2/t)$ for semidiscrete scheme. Subsequently, the results are extended for fully discrete scheme. The error analysis has been carried out on polygonal meshes for discontinuous piecewise polynomials in finite element partitions. Finally, numerical experiments confirm our theoretical convergence results and efficiency of the scheme.

Key words. Parabolic equations, weak Galerkin method, non-smooth data, polygonal mesh, optimal L^2 error estimates.

1. Introduction

There are various applications of parabolic partial differential equations (PDEs) with non-smooth data arising in sciences and engineering such as chemical diffusion, heat conduction processes, thermodynamics, and medical science [6, 7, 14, 16]. Classical finite element methods for parabolic problems with non-smooth initial data have been studied broadly so far, an extensive literature for the same can be obtained from [19, 23–25, 31]. Developing numerical algorithms of parabolic equations with non-smooth initial data has been a flourishing concern. Recently, the weak Galerkin finite element method has attracted much attention in the field of numerical PDEs. As referred to in [28], the weak Galerkin finite element methods (WG-FEMs) have been established as a new finite element technique for solving PDEs, which are derived from weak formulations of problems to replace the classical differential operators (e.g., gradient, divergence, curl) by weak differential operators which is approximated in suitable polynomial spaces and adding the stabilizer term. There is no need to select the parameters of the stabilizer broadly. More precisely, the WG-FEMs have a simple and parameter-free formulation and the flexibility of using general polygonal meshes. With the new concepts of weak function and weak gradient, the WG-FEMs allow discontinuous function space as the approximation space on each element. Unlike the classical finite element method, the WG-FEM is applicable for unstructured polygonal meshes making it more suitable for complex geometry that usually appears in real-life problems. The WG approach has been developed for various types of PDEs in existing literature, such as elliptic equations [18, 21, 26, 29], parabolic equations [9, 10, 17, 33, 34], and the hyperbolic problems [15, 32]. The hybrid high-order (HHO) method is closely

Received by the editors on March 19, 2022; accepted on October 29, 2022.
2020 *Mathematics Subject Classification.* 65N15, 65N30.

related to WG finite element method as the reconstruction operator in the HHO method corresponds to the weak gradient in WG methods [5,11]. The only difference between HHO and WG methods lies in the choice of the discrete unknowns and the stabilization pattern. However, the links between HHO and WG methods are not fully explored yet; nevertheless, they share something in their roots. (cf. [3,8,11]). It is noteworthy that WG and HHO are based on different devising viewpoints and use somewhat different analysis techniques.

We know that the higher order of convergence of finite element approximations depends on the higher smoothness of the true solutions, which demands higher regularity of the initial functions. The main concern of this work is to study the convergence of weak Galerkin finite element approximations for homogeneous equations with non-smooth initial data using polygonal meshes. The error analysis is highly motivated by the fact that the solutions of parabolic problems have the so-called smoothing property (cf. [19]). The solution is smooth for positive time t, even when the initial data are not H^1 regular. Under the low regularity of solutions, convergence analysis has remained a significant part of mathematical study up to the present day. To derive optimal $\mathrm{O}(h^{r+1})$ $(r \geq 1)$ in the L^2 norm for WG-FEM, the minimum regularity assumption on the exact solution u should be $u \in H^1(0,T;H^{r+1}(\Omega))$ (for instance, see [17,34,35]). More recently, in [9], the authors have shown the convergence of WG finite element solution to the true solution at an optimal rate in $L^2(L^2)$ norm under the assumption that $u \in L^2(0,T;H^{r+1}(\Omega)) \cap H^1(0,T;H^{r-1}(\Omega))$. In the case of piecewise linear WG-FEM (i.e., $r=1$), the optimal error estimate requires the initial value to be in H^1 (see, Theorem 3.2 in [9]) and for L^2 initial data error analysis in [9] leads to sub-optimal order of convergence in $L^2(L^2)$ norm (see, Remark 3.2). In fact, optimal $L^\infty(L^2)$ error estimate in [10] for linear weak Galerkin elements demands initial data $u^0 \in H^3(\Omega)$ (see, Remark 3.4). In this work, assuming initial data in L^2, we have shown the convergence of WG finite element solution to the true solution at an optimal rate in L^2 norm on WG finite element space $(\mathcal{P}_1,\ \mathcal{P}_1,\ \mathcal{P}_0^2)$ (see, Theorem 3.2 and Theorem 4.1). The non-smooth data error analysis heavily depends on the newly derived optimal L^2 norm error estimates with smooth initial data $u^0 \in H_0^1 \cap H^2$ (see, Lemma 3.8 and Lemma 4.1). The obtained results intend to enhance the numerical analysis of linear parabolic equations on polygonal meshes with non-smooth initial data. To the best of our knowledge, the smoothing property of the WG-FEM and HHO methods for the parabolic equation has not been studied earlier.

The rest of this work is organized as follows. In Sec. 2, we have introduced some commonly used notations and reviewed the weak Galerkin discretization. Sec. 3 is concerned with the error analysis of the semidiscrete WG finite element algorithm. In Sec. 4, the backward Euler scheme is proposed, and optimal a priori error bounds in $L^\infty(L^2)$ norm is established. Sec. 5 discusses several numerical examples which demonstrate the robustness of