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THE WASSERSTEIN-FISHER-RAO METRIC FOR WAVEFORM BASED EARTHQUAKE LOCATION*

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Abstract

In this paper, we apply the Wasserstein-Fisher-Rao (WFR) metric from the unbalanced optimal transport theory to the earthquake location problem. Compared with the quadratic Wasserstein (W_2) metric from the classical optimal transport theory, the advantage of this method is that it retains the important amplitude information as a new constraint, which avoids the problem of the degeneration of the optimization objective function near the real earthquake hypocenter and origin time. As a result, the deviation of the global minimum of the optimization objective function based on the WFR metric from the true solution can be much smaller than the results based on the W_2 metric when there exists strong data noise. Thus, we develop an accurate earthquake location method under strong data noise. Many numerical experiments verify our conclusions.

Mathematics subject classification: 65K10, 86C08, 86A15, 86A22.
Key words: The Wasserstein-Fisher-Rao metric, The quadratic Wasserstein metric, Inverse theory, Waveform inversion, Earthquake location.

1. Introduction

The optimal transport theory is widely applied in seismology in recent years, leading to more accurate inversion results in the field of geophysical inverse problems, e.g., earthquake location and seismic tomography [11–13, 30, 31, 35, 52, 53]. In these models, appropriate seismic parameters should match the synthetic signals with the observations [36]. From the mathematical point of view, an approximate relationship can be established between seismic parameters $\boldsymbol{m}$ and synthetic seismic signals $\boldsymbol{d}_{syn}(\boldsymbol{m})$ by numerically calculating wave equations. Solving this inverse problem requires $\boldsymbol{m}$ to minimize the difference between the synthetic $\boldsymbol{d}_{syn}(\boldsymbol{m})$ and the observations $\boldsymbol{d}_{obs}$ for a specific metric. The adjoint state method is widely applied to this PDE constrained optimization problem [4–6, 14, 42–44, 50]. The Fréchet gradient of the optimization objective function can be obtained by comparing $\boldsymbol{d}_{obs}$ with $\boldsymbol{d}_{syn}(\boldsymbol{m})$, which is used to update the seismic parameters $\boldsymbol{m}$.

In the past, limited by the computational power, the relationship between parameters $\boldsymbol{m}$ and signals $\boldsymbol{d}_{syn}(\boldsymbol{m})$ was established based on the ray theory [17, 46, 51]. Under this high-frequency

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assumption, some finite frequency phenomena such as wave-front healing and scattering are ignored [42], leading to inaccurate inversion results. With the rapid increase of computational power in recent years, it gradually becomes possible to numerically calculate the wave equation to obtain more accurate synthetic seismic signals $\boldsymbol{d}_{syn}(\boldsymbol{m})$, which mitigates the bias from the high-frequency assumption and raise the inversion resolution reaching the wavelength scale [32, 36, 42, 44, 54, 56, 57].

However, the traditional L_2 norm based waveform inversion suffers from the cycle-skipping problem [26]. Especially for the earthquake location problem, seismic signals are sensitive to the perturbation of the origin time and the earthquake hypocenter. Thus, under the framework of L_2 norm, this point-to-point comparison between signals might generate numerous local minimums, leading to inaccurate inversion results or excessive iteration steps [4, 12]. Based on the optimal transport theory, the Wasserstein metric provides a new perspective to solve these mentioned problems [4, 11, 12, 14, 30, 31, 37, 45, 52, 53]. By comparing signals globally, the Wasserstein metric defined optimization objective function guarantees better convexity property and mitigates the influence of noise. Thus, reasonably accurate inversion results can be expected when the data is contaminated with high-intensity noise [4]. These accurate inversion results could provide significant guidance for the establishment of the early warning system [38], mineral exploration [9], and the siting of major facilities [3].

The quadratic Wasserstein metric requires mass conservation [45]. Thus, the normalization process is mandatory for seismic signals, which becomes an essential limitation. For the earthquake location problem, the amplitude of seismograms provides necessary constraints to the origin time and the distance from the hypocenter to the receiver. Simply normalizing the signals would lead to a nearly flat optimization objective function along a certain direction due to the trade-off between the origin time and the distance. Thus, the minimum point of the optimization objective function may deviate a lot even under the small magnitude of data noise, which leads to low accurate location results, see Examples 3.1 for illustration.

The Wasserstein-Fish-Rao (WFR) metric is a newly developed optimal transport metric and has attracted much attention [7, 8]. This metric is an interpolation between the quadratic Wasserstein metric and the Fisher-Rao metric. From the fluid dynamics point of view [1], this new metric introduces a source term in the continuity equation, allowing the direct comparison between two signals with different total integrals. Thus this metric is also called unbalance optimal transport metric [16, 18, 25, 34], which has been successfully applied in various fields [15, 48, 55]. Benefited from the above features, the normalization of the seismic signals is no longer required. Therefore, the important amplitude information is retained based on the WFR metric, improving the local convexity and avoiding the degeneracy of the optimization objective function near the global minimum point.

Remark 1.1. The Kantorovich-Rubinstein (KR) norm [30, 31] does not require the signals to have the same integral. However, the convexity of the optimization objective functions defined with the KR norm may not be guaranteed for the earthquake location problems. For more details, we refer to see [4], especially Figs. 2–3 for illustration.

Remark 1.2. In [4], we clearly see the superiority of the quadratic Wasserstein metric to the L_2 metric. So in this paper, we prefer to focus on the comparison between the new WFR metric and the quadratic Wasserstein metric.

In this paper, we introduce the WFR metric to the earthquake location problem. It is a significant extension of optimal transport theory in the application to the geophysical inverse