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EXPONENT SPLITTING SCHEMES FOR EVOLUTION EQUATIONS WITH FRACTIONAL POWERS OF OPERATORS

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Abstract. We have considered the Cauchy problem for a first-order evolutionary equation with fractional powers of an operator. Such nonlocal mathematical models are used, for example, to describe anomalous diffusion processes. We want the transition to a new level in time to be solved usual problems. Computational algorithms are constructed based on some approximations of operator functions. Currently, when solving stationary problems with fractional powers of an operator, the most attention is paid to rational approximations. In the approximate solution of nonstationary problems, we come to equations with an additive representation of the problem operator. Additive-operator schemes are constructed by using different variants of splitting schemes. In the present work, the time approximations are based on approximations of the transition operator by the product of exponents. We use exponent splitting schemes of the first and second-order accuracy. The results of numerical experiments for a two-dimensional model problem with fractional powers of the elliptic operator are presented.

Key words. Evolutionary equation, fractional powers of an operator, rational approximation, exponent splitting scheme, stability of operator-difference schemes.

1. Introduction

Spatial effects in classical applied mathematical models (see, for example, [13, 31]) are described most often by elliptic operators of the second order. We relate dynamic processes to parabolic and hyperbolic equations by second-order equations [18, 39]. A characteristic feature of such models is locality in both time and space. The property of locality is manifested in the fact that we write all the terms of the constitutive equations, boundary and initial conditions at the same space-time points.

More and more attention is paid to the study of nonlocal mathematical models. In this regard, we can note the books [14, 17] and the bibliography is given in them. Nonlocality in time is due to the dependence of the system's state studied at a given point in time on the entire prehistory. Such mathematical descriptions of dynamic processes with memory are based on the use of integral and integrodifferential operators [21, 37]. Often (see, e.g., [6, 45]) these nonlocal-time models are associated with fractional derivatives.

Nonlocal models in space are constructed based on various generalizations of classical elliptic operators [7, 15]. In particular, much attention in the literature is paid to fractional Laplacian [16, 36]. Models using elliptic operator functions are involved to describe nonlocal stationary processes. In this case the fractional Laplacian is associated with the fractional degree of the Laplace operator. Local in time and nonlocal in space models are based on abstract parabolic equations with operator functions [57].

Different approaches are used in the approximate solution of stationary problems with operator functions. First of all, we can focus on the achievements of the theory and practice of methods for solving problems with matrix functions

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[20, 26]. For this purpose, the results of a nonlinear approximation of functions [10] are involved, providing an acceptable computational implementations. In [56] methods of rational approximations, approximations by sum and product of exponents with applications to problems with fractional powers of an operator have been distinguished.

We use the spectral definition of the fractional power of a self-adjoint operator in a finite-dimensional Hilbert space. In functional analysis [30] we have a representation based on the Dunford-Taylor integral [28] and the Balakrishnan formula [5]. We note the main approaches for approximate solutions with a fractional power of an operator (see, e.g., [8, 22, 32]) based on rational approximation. The first approach is based on an integral representation using specific quadrature formulas for the Balakrishnan formula [1, 9, 19, 53]. We easily obtain the necessary computational formulas for a rational approximation. The second approach involves using the results of the theory of a best uniform rational approximation of functions [23, 25, 27, 43]. In this case, we can obtain approximations of higher accuracy.

In the approximate solution of nonstationary problems with a fractional power of an operator, we can use standard unconditionally stable implicit time approximations [4, 40]. On the new time level we have a non-standard reaction and fractional diffusion type problems [2, 23, 52]. We can do things a little differently. First, we perform some approximation of a fractional power of an operator, and then we adapt the approximation in time to a peculiarities of the problem. With a rational approximation of a fractional power of an operator, we represent a problem operator as a sum of accessible operators. In this case, we can build time approximations based on the theory of additive operator-difference schemes [33, 48]. Various variants of such splitting schemes are constructed in [12, 51, 55, 54].

In the present paper, the splitting schemes are constructed based on an approximation of the transition operator to a new level in time [34, 48]. For evolution equations of the first order, the transition operator is represented as a product of exponents. For this reason, we call such time approximations exponent splitting schemes. When rational approximations of the fractional power of an operator are used, the operator terms are pairwise permutations. This property allows us to construct exponent splitting schemes most easily.

The paper is organized as follows. In Section 2, the Cauchy problem for the first and second-order evolution equations with a fractional power of a self-adjoint operator in a finite-dimensional Hilbert space is formulated. In Section 3, we discuss the problem of the rational approximations of the fractional power operator for nonstationary problems. The central part Section 4 is devoted to the construction and study of exponent splitting schemes. In Section 5, we address more general evolutionary equations. The results of the numerical solution of the two-dimensional model problem are presented in Section 6. The conclusions follow in Section 7.

2. Problem formulation

After finite-element or finite-volume approximations [29, 38] of elliptic operators, we have discrete analogs in corresponding finite-dimensional spaces. Let H be a finite-dimensional Hilbert space. The scalar product for $u, v \in H$ is (u, v), and the norm is $\|u\| = (u, u)^{1/2}$. For a self-adjoint and positive definite operator D, we define the Hilbert space H_D with scalar product and norm $(u, v)_D = (Du, v)$, $\|u\|_D = (u, v)_D^{1/2}$.

Let A be a self-adjoint positive definite operator ($A : H \to H$):

$$A = A^*, \quad \delta I \le A \le \Delta I, \quad \delta > 0, \tag{1}$$