INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 20, Number 3, Pages 407–436

doi: 10.4208/ijnam2023-1017

A DOUBLY ADAPTIVE PENALTY METHOD FOR THE NAVIER STOKES EQUATIONS

KIERA KEAN, XIHUI XIE, AND SHUXIAN XU

Abstract. We develop, analyze and test adaptive penalty parameter methods. We prove unconditional stability for velocity when adapting the penalty parameter, ϵ, and stability of the velocity time derivative under a condition on the change of the penalty parameter, $\epsilon(t_{n+1}) - \epsilon(t_n)$. The analysis and tests show that adapting $\epsilon(t_{n+1})$ in response to $\nabla \cdot u(t_n)$ removes the problem of picking ϵ and yields good approximations for the velocity. We provide error analysis and numerical tests to support these results. We supplement the adaptive-ϵ method by also adapting the time-step. The penalty parameter ϵ and time-step are adapted independently. We further compare first, second and variable order time-step algorithms. Accurate recovery of pressure remains an open problem.

Key words. Navier-Stokes equations, penalty, adaptive.

1. Introduction

The velocity and pressure of an incompressible, viscous fluid are given by the Navier-Stokes equations. Let $\mathbf{u}$ denote the fluid velocity, p the pressure, ν the kinematic viscosity and f an external force:

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = f, \quad \nabla\cdot\mathbf{u} = 0, \quad \forall(\mathbf{x},t) \in \Omega\times(0,T]. \tag{1}$$

The velocity and pressure are coupled together by the incompressibility constraint $\nabla\cdot\mathbf{u} = 0$. Coupled systems require more memory to store and are more expensive to solve. Penalty methods and artificial compression methods relax the incompressibility condition and result in a pseudo-compressible system. This allows us to uncouple velocity and pressure, which will reduce storage space and computational complexity. Penalty methods that allow complete elimination of the pressure variable are the simplest and fastest, and will be studied herein.

Penalty methods replace $\nabla\cdot\mathbf{u} = 0$ with $\nabla\cdot\mathbf{u} + \epsilon p = 0$ where $0 < \epsilon << 1$. The pressure can be eliminated using $\nabla p = -\nabla(1/\epsilon\nabla\cdot\mathbf{u})$. As the pressure is entirely eliminated from the system, we do not need to solve for it at every timestep, leading to further increases in speed. This method is based on numerical approximation to a single penalty velocity equation, which was introduced by Courant [4]. The accuracy of penalty methods is known to be very sensitive to the choice of ϵ (see Figure 1). This sensitivity suggests considering ϵ a control and picking ϵ through a self-adaptive algorithm. This problem of determining ϵ self-adaptively is considered herein.

When adapting the parameter ϵ, $\|\nabla\cdot\mathbf{u}\|$ is monitored and used to make adjustment to ϵ. The stability of the standard penalty method with variable ϵ is examined in Section 3. No condition on the rate of change of ϵ is required for the stability of $\|\mathbf{u}\|$. However, stability of $\|\mathbf{u}_t\|$ is not unconditional. There is no restriction on the increase of ϵ, however decreasing ϵ quickly will lead to growth in $\|\mathbf{u}\|$. In Section

Received by the editors on July 5, 2022 and, accepted on March 1, 2023.
2000 *Mathematics Subject Classification.* 65M12, 65M60.

3.2, we derive condition (21)

$$(1 - k\alpha)\epsilon_n \le \epsilon_{n+1} \text{ for some } \alpha > 0,$$

where k is the step-size. This condition is required for stability of $\|\mathbf{u}_t\|$. Figure 9 confirms that violating this condition leads to spikes of catastrophic growth in $\|\mathbf{u}_t\|$.

The utility of penalty methods lies in accurate velocity approximation at low cost by simple methods. Consistent with this intent, we couple the adaptive ϵ algorithm with simple, low cost time stepping methods based on the backward Euler method. Simple time filters allow us to implement an effective variable order, variable time-step adaptive scheme, developing further an algorithm of [9]. Graded time steps can resolve the singularity of the solution at $t = 0$ caused by nonsmooth initial data Buyang Li, Shu Ma, and Yuki Ueda [21], Buyang Li, Shu Ma, and Na Wang[20]. The self-adaptive ϵ penalty method can be easily implemented for both constant time-step and variable time-step methods. We develop, analyze and test these new algorithms that independently adapt the time-step k and the penalty parameter ϵ.

In addition to adapting the time-step, we adapt the order of the method between first and second order. This variable time-step variable order (VSVO) method performed better than both first and second-order methods in our tests (see Figure 8 and Figure 9).

The rest of the paper is organized as follows. In Section 2, we introduce important notation and preliminary results. In Section 3, stabilities of $\|\mathbf{u}\|$ and $\|\mathbf{u}_t\|$ for the variable ϵ penalty method with constant timestep is presented. Section 4 presents an error estimate of the semi-discrete, variable ϵ method. Using this we develop an effective algorithm which adapts ϵ and k independently, presented in section 5. We introduce four different algorithms, including the constant time-step and variable time-step variable ϵ method. Numerical tests are shown in Section 6 and open problems are presented in Section 7.

1.1. Review of a Common Penalty Method. Recall the incompressible Navier-Stokes equations, (1). Perturbing the continuity equation by adding a penalty term to the incompressibility condition and explicitly skew-symmetrizing the nonlinear term in the momentum equation in (1) results in the penalty Navier-Stokes equations:

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}(\nabla\cdot\mathbf{u})\mathbf{u} + \nabla p = f, \tag{2}$$

$$\nabla\cdot\mathbf{u} + \epsilon p = 0. \tag{3}$$

By (3), $p = (-1/\epsilon)\nabla\cdot\mathbf{u}$. Inserting this into (2) results in a system of $\mathbf{u}$ only, which is easier to solve than (1):

$$\mathbf{u}_{\epsilon,t} - \nu\Delta\mathbf{u}_\epsilon + \mathbf{u}_\epsilon\cdot\nabla\mathbf{u}_\epsilon + \frac{1}{2}(\nabla\cdot\mathbf{u}_\epsilon)\mathbf{u}_\epsilon - \nabla(\frac{1}{\epsilon}\nabla\cdot\mathbf{u}_\epsilon) = f. \tag{4}$$

From Theorem 1.2 p.120 of Temam [25] we know $\lim_{\epsilon\to 0}(\mathbf{u}_\epsilon(t), p_\epsilon(t)) = (\mathbf{u}(t), p(t))$.

Consider the first-order discretization of (2)-(3). k_n is the nth time-step, ϵ_n is the parameter ϵ at nth time-step, $t_0 = 0, t_n = t_{n-1} + k_n$. Let $\mathbf{u}^*$ denote the standard (second order) linear extrapolation of $\mathbf{u}$ at t_{n+1}:

$$\mathbf{u}^* = \left(1 + \frac{k_{n+1}}{k_n}\right)\mathbf{u}^n - \frac{k_{n+1}}{k_n}\mathbf{u}^{n-1}.$$