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New Rational Interpolation Basis Functions on the Unbounded Intervals and Their Applications

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Abstract. Based on the rational system of Legendre rational functions, we construct two set of new interpolation basis functions on the unbounded intervals. Their explicit expressions are derived, and fast and stable algorithms are provided for computing the new basis functions. As applications, new rational collocation methods based on these new basis functions are proposed for solving various second-order differential equations on the unbounded domains. Numerical experiments illustrate that our new methods are more effective and stable than the existing collocation methods.

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Key words: Rational collocation methods, Birkhoff interpolation, fast and stable algorithms.

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1. Introduction

Many problems arising in quantum mechanics, astrophysics, fluid dynamics and other fields can be described by differential equations on unbounded domains, see [1, 3, 5, 14, 15, 19, 21, 24, 25] and the references therein.

The classical Legendre rational spectral method on the half line (cf. [27] with $\alpha = 0, \beta = 0$) is based on the basis functions $\{L_k((x-1)/(x+1))\}$, where $L_k(\cdot)$ is the Legendre polynomial of degree k, which are mutually orthogonal with the weight function $\omega(x) := (x+1)^{-2}$. Using the rational spectral method, differential equations whose solutions are exponential or algebraic decay can be effectively approximated. However, the symmetry and conservation of the problems may be destroyed by the

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weight function $\omega(x)$. Furthermore, the inclusion of $\omega(x)$ also makes the theoretical analysis more difficult. On the other way, one can directly use the Laguerre functions

$$\widehat{\mathcal{L}}_k^{(\alpha,\beta)}(x) := \frac{1}{k!} x^{-\alpha} e^{\frac{\beta}{2}x} \partial_x^k (x^{k+\alpha} e^{-\beta x}), \quad \alpha > -1, \quad \beta > 0$$

as the basis functions, which are mutually orthogonal with respect to the weight function x^α. Unfortunately, the approximation effects for algebraic decay functions are not good using this method, which may greatly limit the practical applications. In order to combine the respective advantages of the two methods, Gu and Wang [17] introduced a new family of Jacobi rational functions $\widehat{R}_n^{(\gamma,\alpha)}(x)$, which is mutually orthogonal with respect to the weight function x^α on the half line, and its weight function is the same as that of the Laguerre functions. Using the new Jacobi rational basis functions, the symmetry and conservation of the schemes can be guaranteed by selecting the parameter α. Moreover, the exponential or algebraic decay functions can be effectively approximated and the convergence analysis of the new Jacobi rational spectral methods becomes easier. Similarly, for a problem on the whole line, we can use another rational mapping $h(x) = x/\sqrt{x^2+1}$ which has a better approximation effect for the algebraic decay functions, see [2, 26, 30, 32].

For the nonlinear and variable coefficient problems, the spectral collocation method which performs differentiations on a set of preassigned collocation points has remarkable advantage compared with the spectral-Galerkin method. However, the ill-conditioned system produced by this method has an impact on the numerical stability. To circumvent this barrier, the construction of suitable preconditioners is a significant approach. Effective attempts are available, such as low-order finite difference or finite elements [4, 6, 10, 11, 22, 23], and preconditioning differentiation by integration [7–9, 12, 13, 16, 20, 28, 29, 31]. In addition, suitable basis functions can be built from Birkhoff interpolation, and usually provide good appropriate preconditioners for usual collocation schemes, see [18, 28, 33, 34].

The purpose of this paper is to extend the work of [17, 27, 28], in order to obtain more suitable rational collocation methods based on Birkhoff interpolation for solving differential equations whose solutions decay at the infinity. To this end, we construct new interpolation basis functions based on Legendre rational basis functions on the unbounded intervals by Birkhoff interpolation problems. We exhibit the new rational basis functions in closed form and also provide fast and stable algorithms to compute them using the relationship between Birkhoff interpolation basis functions on the bounded intervals and the Legendre rational interpolation basis functions. Furthermore, we propose new rational collocation methods based on the proposed basis functions for various second-order problems on the unbounded domains. Numerical results are provided to show the advantages of the new approaches.

The paper is organized as follows. In Section 2, we first construct a new set of rational interpolation basis functions on the half line and provide an efficient and stable algorithm to compute them. Various problems on the semi-unbounded domains are considered and supply with numerical results to show higher accuracy and better stabil-