East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2022-204.241022

Vol. **13**, No. 2, pp. 361-397
May 2023

Stability and Convergence of Stabilized Finite Volume Iterative Methods for Steady Incompressible MHD Flows with Different Viscosities

Xiaochen Chu, Chuanjun Chen and Tong Zhang*

School of Mathematics and Information Sciences, Yantai University, Yantai, 264005, China.

Received 7 July 2022; Accepted (in revised version) 24 October 2022.

Abstract. Three finite volume iterative schemes for steady incompressible magnetohydrodynamic problems are studied. The theoretical analysis of finite volume methods is more challenging than of finite element methods because of the presence of a trilinear form and the difficulties with the treatment of nonlinear terms. Nevertheless, we prove the uniform stability of the methods and establish error estimates. It is worth noting that the Newton iterative scheme converges exponentially under viscosity related requirements, while the Oseen iterative method is unconditionally stable and convergent under the uniqueness condition. Nome numerical examples confirm the theoretical findings and demonstrate a good performance of the methods under consideration.

AMS subject classifications: 65M10, 65N30, 76Q10

Key words: Incompressible MHD equation, finite volume method, L^2-projection, iterative scheme.

1. Introduction

The finite difference, finite volume and finite element methods are the three main numerical approaches solving partial differential equations. The finite volume method (FVM) retains many advantages of the finite difference method (FD) and the finite element method (FEM), such as the easiness of the construction and implementation, the local mass conservation, and the flexibility in handling complex geometric regions and boundary conditions in practical applications. As a result, the FVM is widely used in scientific and engineering computations. For example, we can refer the reader to [2,4,6,13,14,46,53,54] for elliptic problems, [5,9,15,18,32,43,55] for parabolic problems, [21,25,26,33,35,42,47,51] for incompressible Stokes and Navier-Stokes equations.

The incompressible magnetohydrodynamics (MHD) problem is a system describing the behavior of a fluid in magnetic field and the interaction of conducting fluids and magnetic

*Corresponding author. *Email addresses:* `cjchen@ytu.edu.cn` (C. Chen), `tzhang@ytu.edu.cn` (T. Zhang)

 ©2023 Global-Science Press

fields. It has been developed and applied in plasma physics, controlled thermonuclear fusion, astrophysical research, and electromagnetic propulsion. In this paper, we consider the following steady incompressible MHD equations:

$$\begin{aligned} &-R_e^{-1}\Delta u + u\cdot\nabla u + \nabla p - S_c\nabla\times B\times B = f, && x\in\Omega,\\ &S_cR_m^{-1}\nabla\times(\nabla\times B) - S_c\nabla\times(u\times B) = g, && x\in\Omega,\\ &\nabla\cdot u = 0, && x\in\Omega,\\ &\nabla\cdot B = 0, && x\in\Omega \end{aligned} \tag{1.1}$$

with the homogeneous boundary conditions

$$\begin{aligned} &u|_{\partial\Omega} = 0, && \text{(no-slip condition)},\\ &(B\cdot n)|_{\partial\Omega} = 0, \quad (n\times\nabla\times B)|_{\partial\Omega} = 0, && \text{(perfectly wall)}, \end{aligned} \tag{1.2}$$

where $\Omega\subset\mathbb{R}^d$, $d=2,3$ is a bounded polygonal/polyhedral domain with the boundary $\partial\Omega$ satisfying Condition 2.1 below, u the velocity, p the pressure, and B the magnetic field. Besides, R_e, R_m and S_c are the hydrodynamic Reynolds number, magnetic Reynolds number and coupling number, whereas n refers to the outer unit normal to $\partial\Omega$, and f, g are external forces with g satisfying the compatibility condition $\nabla\cdot g = 0$.

Numerical methods are important tools in studying the MHD problem (1.1)-(1.2), since it is not possible to find its analytic solutions because of the incompressibility, nonlinearity and the multi-variable coupling. In particular, Gunzbunger *et al.* [20] proved the existence and uniqueness of Galerkin finite element solutions and established optimal error estimates, Greif *et al.* [19] obtained optimal error estimates for the discontinuous Galerkin method, Zhang *et al.* [48] provided the uniform stability and convergence of the streamline diffusion method. After that, interesting results have been obtained for two-level iterative finite element methods [11, 12, 40], penalty iterative methods [36–38], nonconforming finite element methods [31,34], correction schemes [39,49,50], and unconditional energy stable schemes [7,8].

From the above papers and references, we note that almost all the existed numerical results are related to finite element methods. Motivated by advantages of finite volume methods and inspired by the works [23, 26, 29, 33] about the stabilized FVM for incompressible Navier-Stokes equations, we consider the stability and convergence of stabilized finite volume iterative methods for the steady incompressible MHD equations (1.1). Unlike to the finite element method for the problem (1.1), there are two major difficulties in theoretical analysis. One is the relationship of bilinear terms in FEM and FVM with the conforming finite elements of the lowest-order pair. In particular, the treatment of the term $\nabla\times(\nabla\times B)$ in finite volume methods is a key issue. The other major difficulty is the treatment of nonlinear terms — e.g. trilinear ones, arising in the finite volume schemes for incompressible MHD equations, where the weak form of $u\cdot\nabla u$ is not anti-symmetric as in FEMs [41, 44]. Besides, the other nonlinear terms $\nabla\times B\times B$ and $\nabla\times(u\times B)$ have also not been considered in finite volume methods before. Hence, for the incompressible MHD equations the theoretical analysis of finite volume methods is more challenging than of finite element methods.