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A Hybrid Method for Three-Dimensional Semi-Linear Elliptic Equations

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Abstract. This paper is concerned with a hybrid method for three-dimensional semi-linear elliptic equations, constructed by combining the ideas presented in [Huang *et al.*, J. Comput. Phys. **419** (2020)] and [Zhang *et al.*, Comput. Math. Appl. **80** (2020)]. The convergence rate analysis indicates that the method converges rapidly. Numerical examples support the theoretical results and show that the method proposed outperforms the purely deep learning-based and traditional iterative methods.

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Key words: Deep learning, iterative method, semi-linear elliptic equations, convergence rate analysis.

1. Introduction

The deep learning technique experiences a great success in pattern recognition, computer vision, and natural language processing [9]. Such an approach is also widely used in computational and applied mathematics, and various artificial neural networks based numerical methods for partial differential equations (PDEs) have been developed in the past few years — cf. Refs. [8, 10, 19, 20, 25]. Unlike the traditional methods, most of the deep learning approaches are mesh-free and very efficient in handling numerous high-dimensional problems [6]. In particular, the deep neural networks (DNNs) are adopted to parameterize the solutions of PDEs and determine the relevant parameters by solving an optimization problem for the PDE concerned. The efficiency of this approach relies on the strong expressive power of deep neural networks [7,11,17]. The reader can consult [5] for a very comprehensive review of the machine learning from a perspective of computational mathematics.

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On the other hand, deep learning methods still have own limitations — e.g., usually, the number of iterations is very large, typically $\mathcal{O}(10000)$, and the accuracy is very low, typically about 10^{-2} to 10^{-4}. Taking into account these observations, Huang *et al.* [13] introduced an Int-Deep method for nonlinear variational problems consisting of two phases. In the first phase, the underlying problem is solved by a deep learning method. The solution obtained is used as the initial guess in the second phase. We note that the Newton's method or its variants invoked to solve the nonlinear systems arising in the finite element discretization of the variation problems, quickly deliver a highly efficient numerical solution. This method has been applied to semi-linear elliptic problems in one- and two-dimensional cases and to eigenvalue problems. The error analysis has been also carried out. The rationality of the method can be interpreted intuitively based on the frequency principle [24] or the spectral bias [18]. The frequency principle states that a deep learning method is tempted to capture low-frequency components of the exact solution in advance during the learning process. This implies that the deep learning solution can be viewed as a numerical solution related to a coarse discretization of the variational problem. Therefore, it can be used as an initial guess when solving discrete problems over fine meshes.

In this work, we develop a hybrid iterative method for three-dimensional semi-linear elliptic problems by combining the ideas in [13, 26]. Using the recursive analysis, we study the convergence of the method under the assumptions similar to [26]. Theoretical analysis and numerical simulations show that the hybrid method outperforms purely deep learning-based methods and traditional iterative methods. It is worth noting that the method converges rapidly and convergence rates obtained are different from [26]. More exactly, if E_k is the difference between the finite element solution and the k-th solution of the hybrid algorithm, then using the Sobolev embedding theorem, the Poincaré-Friedrichs inequality, and the Hölder inequality, we show that

$$\left\|v_k^h\right\|_1 \lesssim \|E_k\|_1, \quad \left\|E_k - v_k^h\right\|_1 \lesssim \|E_k\|_0 \|E_k\|_1,$$

where v_k^h denotes the k-th update of Newton's method (3.6), cf. Lemmas 4.2 and 4.3. The last two estimates yield the inequality

$$\|E_{k+1}\|_1 \lesssim \|E_k\|_1^2,$$

and the result desired — cf. Lemma 4.4. Here and in what follows, we write $a \lesssim b$ if $a \leq Cb$ with the universal positive constant C independent of the finite element mesh size h and the iteration step k. The constant C may take different values at different occurrences. Besides, we denote by $(\cdot,\cdot)$ the standard L^2-inner product over Ω and use standard notation for Sobolev spaces, norms, and semi-norms [1].

The rest of this paper is organized as follows. In Section 2, we introduce the DNN which will be used later on. Section 3 contains the description of a hybrid method for semi-linear elliptic equations. The convergence rate of the method are studied in Section 4. Two numerical examples in Section 5 demonstrate the performance of the method under consideration.