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FULL DISCRETISATION OF THE TIME DEPENDENT NAVIER-STOKES EQUATIONS WITH ANISOTROPIC SLIP BOUNDARY CONDITION

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Abstract. In this work, we study theoretically and numerically the non-stationary Navier-Stokes's equations under power law slip boundary condition. We establish existence of a unique solution by using a semi-discretization in time combined with the weak convergence approach. Next, we formulate and analyze the discretzation in time and the finite element approximation in space associated to the continuous problem. We derive optimal convergence in time and space provided that the solution is regular enough on the slip zone. Iterative schemes for solving the nonlinear problems is formulated and convergence is studied. Numerical experiments presented confirm the theoretical findings.

Key words. Power law slip boundary condition, Navier-Stokes equations, space-time discretization, monotonicity, error estimates.

1. Introduction

We are concerned with the dicretization of the non-stationary incompressible Navier-Stokes equations

$$(1)\qquad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - 2\nu\,\mathrm{div} D\mathbf{u} + [\mathbf{u}\cdot\nabla]\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega\times(0,T),\\ \mathrm{div}\,\mathbf{u} &= 0 \quad \text{in } \Omega\times(0,T)\\ \mathbf{u}(\mathbf{x},\mathbf{0}) &= \mathbf{u}_0 \quad \text{in } \Omega\times\{0\}, \end{cases}$$

where Ω is a open and bounded domain in $\mathbb{R}^d$, with a Lipschitz-continuous boundary $\partial\Omega$. It is assumed that $d = 2, 3$, and $T > 0$ is the final time of observation of the fluid. The unknowns are the velocity $\mathbf{u}$ and the pressure p. $\mathbf{f}$ is the external force acting on the fluid and ν is the kinematic viscosity of the fluid, assume non-negative. $\mathbf{u}_0$ is the initial velocity and we assume for the moment that div $\mathbf{u}_0 = 0$. We recall that the Cauchy stress tensor is $\mathbf{T} = -p\mathbf{I} + 2\nu D\mathbf{u}$, with $\mathbf{I}$, the identity matrix in $\mathbb{R}^{d\times d}$, while the symmetric part of the velocity gradient is $2D\mathbf{u} = \nabla\mathbf{u} + (\nabla\mathbf{u})^T$. We are interested in (1) when the position and the direction of the slip boundary condition are taken into account (see [14, 15]). We then assume that the boundary $\partial\Omega$ is made of two components S and Γ, such that $\overline{\partial\Omega} = \overline{S\cup\Gamma}$, with $S\cap\Gamma = \emptyset$. We assume the homogeneous Dirichlet condition on Γ, that is

$$(2)\qquad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma\ .$$

Thus Γ is the porous or artificial boundary where the fluid is prescribed. On S, we assume the impermeability condition

$$(3)\qquad \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } S\,,$$

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where $\mathbf{n} : S \longrightarrow \mathbb{R}^d$ is the normal outward unit vector to S. S is an impermeable solid surface along which the fluid may slip. Taking the scalar product of $\mathbf{u}$ and the balance of linear momentum in (1), we obtain

(4)
$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathbf{u}|^2\,dx + 2\nu\int_\Omega |D\mathbf{u}|^2\,dx + \int_S (-\mathbf{Tn})_{\boldsymbol{\tau}}\cdot\mathbf{u}_{\boldsymbol{\tau}}d\sigma = \int_\Omega \mathbf{f}\cdot\mathbf{u}dx\ , \quad \text{for all } t \geq 0\,,$$

with $d\sigma$ being the surface measure associated to S. Also, for any vector $\mathbf{v}$ defined on S, we set $\mathbf{v}_{\boldsymbol{\tau}} = \mathbf{v} - (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$. Thus $(\mathbf{Tn})_{\boldsymbol{\tau}}$ denotes the projection of the normal stress into the corresponding tangent plane. We note that the first term in (4) stands for the change of the kinetic energy, the second and third expressions represent the energy that is dissipated and transformed to other form forms of energy. We are more interested in the energy on the boundary S that can only be fully expressed if $(\mathbf{Tn})_{\boldsymbol{\tau}}$ is given. For that purpose, the most general relation between $\mathbf{u}_{\boldsymbol{\tau}}$ and $(\mathbf{Tn})_{\boldsymbol{\tau}}$ is the implicit constitutive relation [19]

(5)
$$\psi(\mathbf{u}_{\boldsymbol{\tau}}, (\mathbf{Tn})_{\boldsymbol{\tau}}) = 0$$

where ψ is function. The simplest form of (5) that ensure the non-negativity of $\displaystyle\int_S (-\mathbf{Tn})_{\boldsymbol{\tau}}\cdot\mathbf{u}_{\boldsymbol{\tau}}d\sigma$ is the choice

$$(-\mathbf{Tn})_{\boldsymbol{\tau}} = \alpha\mathbf{u}_{\boldsymbol{\tau}}d\sigma \quad , \quad \alpha > 0\ .$$

This is the Navier's slip boundary conditions. If $(\mathbf{Tn})_{\boldsymbol{\tau}} = \mathbf{0}$, then one gets a perfect slip boundary condition, while if $\mathbf{u}_{\boldsymbol{\tau}} = \mathbf{0}$, then there is no slip. We are interested in the power law slip boundary condition given as follows [7]

(6)
$$(\mathbf{Tn})_{\boldsymbol{\tau}} + |K\mathbf{u}_{\boldsymbol{\tau}}|^{s-2}K^2\mathbf{u}_{\boldsymbol{\tau}} = 0 \quad \text{on } S\times(0,T)\,,$$

where $|\mathbf{v}|^2 = \mathbf{v}\cdot\mathbf{v}$ is the Euclidean norm. K is an anisotropic tensor, assumed to be uniformly positive definite, symmetric, and bounded. s is a real, strictly positive number representing the flow behavior index. The tangential shear is a power law function of the tangential velocity. Such a boundary condition arises when the contact surface is lubricated with a thin layer of a non-Newtonian fluid. It is manifest that for $s = 2$ and $K = \mathbf{I}$, one obtains the classical Navier's slip condition. The anisotropic slip law (6) defines from the slip relation introduce in [14, 15]; that is

(7)
$$(\mathbf{Tn})_{\boldsymbol{\tau}} + \psi(\mathbf{u}_{\boldsymbol{\tau}})\mathbf{u}_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{on } S\times(0,T)\,,$$

where the function ψ is real valued and satisfies;

(i) ψ is bounded and there exist two positive constants α_1, α_2 such that for any vector $\mathbf{v}\in\mathbb{R}^d$

(8)
$$\alpha_1 \leq \psi(\mathbf{v}) \leq \alpha_2\ .$$

(ii) ψ is Lipschitz-continuous with Lipschitz constant λ, that is

(9)
$$\forall\ \mathbf{v}, \boldsymbol{w}\in\mathbb{R}^d\ ,\ |\psi(\mathbf{v}) - \psi(\boldsymbol{w})| \leq \lambda\,|\mathbf{v} - \boldsymbol{w}|\ .$$

It is manifest by taking $\psi(\mathbf{u}_{\boldsymbol{\tau}}) = |K\mathbf{u}_{\boldsymbol{\tau}}|^{s-2}K^2$, the conditions (8) and (9) are not verified. Hence (6) does not belongs to the class of anisotropic slip boundary conditions defined by C. Le Roux in [14, 15]. We intend to study the finite element solution of the Navier-Stokes equations with (7), (8) and (9). A similar model but for the stationary case has been analysed in [7] using conforming finite element approach.