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A SPLITTING SPECTRAL METHOD FOR THE NONLINEAR DIRAC-POISSON EQUATIONS

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Abstract. We develop a splitting spectral method for the time-dependent nonlinear Dirac-Poisson (DP) equations. Through time splitting method, we split the time-dependent nonlinear DP equations into linear and nonlinear subproblems. To advance DP from time t_n to t_{n+1}, the nonlinear subproblem can be integrated analytically, and linear Dirac and Poisson equation are well resolved by Fourier and Sine spectral method respectively. Compared with conventional numerical methods, our method achieves spectral accuracy in space, conserves total charge on the discrete level. Extensive numerical results confirm the spatial spectral accuracy, the second order temporal accuracy, and the l^2-stable property. Finally, an application from laser field is proposed to simulate the spin-flip phenomenon.

Key words. Nonlinear Dirac-Poisson equations, spectral method, splitting method, laser field, spin-flip.

1. Introduction

Maxwell-Dirac (MD) system represents the time-evolution of fast (relativistic) electrons and positrons within external and self-consistent generated electromagnetic fields, and it plays an important role in quantum electrodynamics [20, 21]. The system combining Maxwell equations and Dirac equations is of great significance to the progress of science and technology, the rapid development of information age. And it has been widely employed in many areas such as quantum cosmology, atomic physics, nuclear physics, gravitational physics [10, 23, 26].

Under the electrostatic condition, the Dirac-Poisson (DP) system can be directly derived from the MD system and it also can be adopted to study theoretically the structures and/or dynamical properties of materials. In 1966, Wakano obtained the localized solutions of the MD system under the electrostatic field [27]; in 1976, Chadam and Glassey studied the solution of the 2d MD system with zero magnetic field [9]; in 1994, Esteban and Sere confirmed the existence of stationary solutions for the DP system [13]; in 2014, Brinkman, Heitzinger and Markowich used the DP system to simulate graphene [7]; in 2017, Zhang et al. used variational methods to analyze the existence of infinitely many stationary solutions for the DP system [31].

Up to now, there are a few numerical methods to solve nonlinear Dirac equations: Finite difference methods [7, 17, 24], Runge-Kutta discontinuous Galerkin methods[25, 29, 30], Fourier spectral methods [1, 3, 4, 6, 16], while efficient high-order numerical methods for nonlinear DP equations are scarce.

In this paper, we propose a novel splitting spectral method for the time-dependent DP equations. In time, we apply the splitting method. In space, we apply Fourier spectral method and Sine spectral method to discretize the Dirac and Poisson equations, respectively. The merits of the proposed method for the nonlinear DP equations are that it is unconditionally stable, fast in computation,

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has spectral accuracy in space, and conserves the particle number, momentum and energy of the system.

The organization of the article is following. In section 2, we introduce the nonlinear DP equations and its dimensionless formulation. Besides, we give the definition of charge, momentum, energy and prove the conservation laws. In section 3, we propose a splitting spectral method for the time-dependent DP equations and introduce the detailed numerical algorithm in time and space, respectively. In section 4, we analyze the stability and convergence of the splitting spectral method. In section 5, we present numerical tests in 1-d, 2-d and 3-d DP equations, respectively. As an application, we consider the laser-atom dynamics with the 2-d DP equations. Finally, conclusion is drawn.

2. The nonlinear DP equations

2.1. Introduction of the nonlinear DP equations. MD system has been studied widely in quantum electrodynamics, and in this paper, we consider the following electrostatic MD equations

$$i\hbar\partial_t\boldsymbol{\Psi} = (-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + e\boldsymbol{\alpha}\cdot\boldsymbol{A} + mc^2\boldsymbol{\beta})\boldsymbol{\Psi} - e\phi\boldsymbol{\Psi}, \tag{1}$$

$$-\boldsymbol{\nabla}^2\phi = |e||\boldsymbol{\Psi}|^2. \tag{2}$$

In [12, 18], the equations are called Dirac-Poisson equations. And c is the speed of light, e is the elementary charge, m is the electron or positron mass, $\hbar$ is the Planck's constant, i is the imaginary unit, $\boldsymbol{A}(\boldsymbol{x},t) = (A_1(\boldsymbol{x},t), A_2(\boldsymbol{x},t), A_3(\boldsymbol{x},t))^T$ are the electromagnetic vector potentials, $\phi(\boldsymbol{x},t)$ is the electric potential. And the unknown $\boldsymbol{\Psi}$ is the 4-vector complex wave function of the 'spinorfield': $\boldsymbol{\Psi}(\boldsymbol{x},t) = (\Psi_1(\boldsymbol{x},t), \Psi_2(\boldsymbol{x},t), \Psi_3(\boldsymbol{x},t), \Psi_4(\boldsymbol{x},t))^T$, $\boldsymbol{x} = (x_1, x_2, x_3)^T$ is the spatial coordinates, $\boldsymbol{\nabla} = (\partial_1, \partial_2, \partial_3)^T$, and $\boldsymbol{\nabla}^2 = \partial_{11}^2 + \partial_{22}^2 + \partial_{33}^2$. Explicitly, $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3)^T$ and $\boldsymbol{\beta}$ are the 4×4 Pauli-Dirac matrices

$$\boldsymbol{\beta} = \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{I} \end{pmatrix}, \quad \boldsymbol{\alpha}_\eta = \begin{pmatrix} \boldsymbol{0} & \boldsymbol{\sigma}_\eta \\ \boldsymbol{\sigma}_\eta & \boldsymbol{0} \end{pmatrix}, \quad \eta = 1,2,3,$$

where $\boldsymbol{I}, \boldsymbol{0}$ and $\boldsymbol{\sigma}_\eta(\eta = 1,2,3)$ are the 2×2 identity matrix, null matrix and Pauli matrices, respectively. i.e.

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Multiplying both-hand sides of the Eq.(1) by $\frac{-i}{mc^2}$ to quantify it, we have

$$\frac{\hbar}{mc^2}\partial_t\boldsymbol{\Psi} = (-\frac{\hbar}{mc}\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \frac{e}{mc^2}i\boldsymbol{\alpha}\cdot\boldsymbol{A} - i\boldsymbol{\beta})\boldsymbol{\Psi} + \frac{e}{mc^2}i\phi\boldsymbol{\Psi}. \tag{3}$$

Now, we rescale the space, the time, the wave function and the potential function by setting

$$\begin{gathered} t = \tfrac{\hbar}{mc^2}\tilde{t}, \quad \boldsymbol{x} = \tfrac{\hbar}{mc}\tilde{\boldsymbol{x}}, \quad \boldsymbol{A} = \tfrac{e}{mc^2}\tilde{\boldsymbol{A}}, \quad \phi = \tfrac{e}{mc^2}\tilde{\phi}, \\ \boldsymbol{\Psi} = (\tfrac{\hbar}{mc})^{3/2}\tilde{\boldsymbol{\Psi}}, \quad \tilde{\boldsymbol{\Psi}} = \tilde{\boldsymbol{\Psi}}(\tilde{\boldsymbol{x}}, \tilde{t}), \quad \tilde{\boldsymbol{x}} \in \mathbb{R}^3. \end{gathered} \tag{4}$$

Substituting Eq.(4) into Eq.(3) and Eq.(2), then moving all $\sim$, we get the following dimensionless DP equations in 3-d

$$\begin{gathered} \partial_t\boldsymbol{\Psi} = (-\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - i\boldsymbol{\alpha}\cdot\boldsymbol{A} - i\boldsymbol{\beta})\boldsymbol{\Psi} + i\phi\boldsymbol{\Psi}, \\ \boldsymbol{\nabla}^2\phi = -\lambda^2|\boldsymbol{\Psi}|^2. \end{gathered} \tag{5}$$

where the coefficient of the Poisson equation depends on λ^2 which is related to the scaling value for wave function $\boldsymbol{\Psi}$.