East Asian Journal on Applied Mathematics
doi: 10.4208/eajam.2022-271.090123

Vol. **13**, No. 3, pp. 464-498
August 2023

# Stability of Spectral Collocation Schemes with Explicit-Implicit-Null Time-Marching for Convection-Diffusion and Convection-Dispersion Equations

Meiqi Tan[1], Juan Cheng[2,3,*] and Chi-Wang Shu[4]

[1] *Graduate School, China Academy of Engineering Physics, Beijing 100088, China.*
[2] *Laboratory of Computational Physics, Institute of Applied Physics and Computational Mathematics, Beijing 100088, China.*
[3] *HEDPS, Center for Applied Physics and Technology, and College of Engineering, Peking University, Beijing 100871, China.*
[4] *Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.*

*Received 3 October 2022; Accepted (in revised version) 9 January 2023.*

Dedicated to Professor Tao Tang on the occasion of his 60th birthday.

---

**Abstract.** In this paper, we discuss the Fourier collocation and Chebyshev collocation schemes coupled with two specific high order explicit-implicit-null (EIN) time-marching methods for solving the convection-diffusion and convection-dispersion equations. The basic idea of the EIN method discussed in this paper is to add and subtract an appropriate large linear highest derivative term on one side of the considered equation, and then apply the implicit-explicit time-marching method to the equivalent equation. The EIN method so designed does not need any nonlinear iterative solver, and the severe time step restriction for explicit methods can be removed. We give stability analysis for the proposed EIN Fourier collocation schemes on simplified linear equations by the aid of the Fourier method. We show rigorously that the resulting schemes are stable with particular emphasis on the use of large time steps if appropriate stabilization parameters are chosen. Even though the analysis is only performed on the EIN Fourier collocation schemes, numerical results show that the stability criteria can also be extended to the EIN Chebyshev collocation schemes. Numerical experiments are given to demonstrate the stability, accuracy and performance of the EIN schemes for both one-dimensional and two-dimensional linear and nonlinear equations.

**AMS subject classifications**: 65M70, 65M12

**Key words**: Convection-diffusion equation, convection-dispersion equation, stability, explicit-implicit-null time discretization, spectral collocation method.

---

*Corresponding author. *Email addresses:* `tanmeiqi20@gscaep.ac.cn` (M. Tan), `cheng_juan@iapcm.ac.cn` (J. Cheng), `chi-wang_shu@brown.edu` (C.-W. Shu)

 

## 1. Introduction

In this paper, we will discuss the Fourier collocation and Chebyshev collocation methods coupled with two different explicit-implicit-null (EIN) time-marching methods for the convection-diffusion and convection-dispersion equations, respectively, with an eye to basic questions of accuracy and stability of the schemes. We restrict the description to problems in one dimension for symbolic simplicity, although the conclusions are verified to hold also for two-dimensional equations in the numerical experiment sections.

The convection-diffusion equation

$$U_t + f(U)_x - \mathscr{D}(U)_{xx} = 0, \quad \mathscr{D}(U) = \int^U d(s)\,\mathrm{d}s, \tag{1.1}$$

where $d(s) \geq 0$ is smooth and bounded, has been used in many areas of science and technology; e.g., fluid dynamics, heat transfer and environmental protection. For an extensive literature devoted to the above equation let us mention the papers [21, 40], and the references therein. Here and below, we use the capital letter $U$ to denote the exact solution of the considered equation.

The convection-dispersion equation

$$U_t + f(U)_x + \mathscr{G}(U_x)_{xx} = 0 \tag{1.2}$$

is a special KdV-type equation typified by the Korteweg-de Vries (KdV) equation [26] and its generalizations. The KdV-type equations, whose travelling-wave solutions called solitary waves play an important role in the long-term evolution of initial data [6], have especially important applications as a widely used model of nonlinear dispersion in fluid dynamics and plasma physics.

For the above two equations, the Fourier collocation method is a popular numerical method, which grants the use of the fast Fourier transform. However, a disadvantage of the use of the Fourier basis is the confinement to periodic boundary condition. In some situations, one may want to consider problems involving non-periodic boundary conditions. In this case, we can turn to the pseudo-spectral Chebyshev method, i.e., a collocation method at the Chebyshev Gauss nodes. There is an extensive body of bibliography [1, 7, 8, 13, 23, 34] on the numerical simulation and analysis of the convection-diffusion and convection-dispersion equations in conjunction with the Fourier collocation or Chebyshev collocation method for spatial discretization. We refer to [1, 13, 34] for the convection-dispersion equation, and to [7, 8, 23] for the convection-diffusion equation. Limited by the time-marching method, to our knowledge, no high order numerical schemes can efficiently simulate the above two kinds of time-dependent equations with nonlinear highest derivative terms at large time steps whilst keeping stability, especially when the Chebyshev collocation method is used for spatial discretization.

Time discretization is a very important issue for time-dependent partial differential equations (PDE). The explicit time-marching methods are easy to implement, however, they become unfeasible with growing spatial order due to the worsening stiffness of the