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A Well-Balanced Partial Relaxation Scheme for the Two-Dimensional Saint-Venant System

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Abstract. We develop a new moving-water equilibria preserving partial relaxation (PR) scheme for the two-dimensional (2-D) Saint-Venant system of shallow water equations. The new scheme is a 2-D generalization of the one-dimensional (1-D) PR scheme recently proposed in [X. Liu, X. Chen, S. Jin, A. Kurganov, and H. Yu, SIAM J. Sci. Comput., 42 (2020), pp. A2206–A2229]. Our scheme is based on the PR approximation, which is designed in two steps. First, the geometric source terms are incorporated into the discharge fluxes, which results in a hyperbolic system with global fluxes. Second, the discharge equations are relaxed so that the nonlinearity is moved into the stiff right-hand side of the four added auxiliary equation. The obtained PR system is then numerically integrated using a semi-discrete hybrid upwind/central-upwind finite-volume method combined with an efficient semi-implicit ODE solver. The new 2-D PR scheme inherits the main advantages of the 1-D PR scheme: (i) no special treatment of the geometric source terms is required, (ii) no nonlinear (cubic) equations should be solved to obtain the point values of the water depth out of the reconstructed equilibrium variables. The performance of the proposed PR scheme is illustrated on a number of numerical examples, in which we demonstrate that the PR scheme not only capable of exactly preserving quasi 1-D moving-water steady states and accurately capturing their small perturbations, but can also handle genuinely 2-D steady states and their small perturbations in a non-oscillatory manner.

AMS subject classifications: 76M12, 65M08, 86-08, 86A05, 35L65

Key words: Saint-Venant system of shallow water equations, partial relaxation scheme, well-balanced method, steady-state solutions (equilibria), moving-water and still-water equilibria.

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1 Introduction

We consider the two-dimensional (2-D) Saint-Venant system of shallow water equations:

$$\begin{cases} h_t+q_x+p_y=0, \\ q_t+\Big(hu^2+\dfrac{g}{2}h^2\Big)_x+(huv)_y=-ghB_x, \\ p_t+(huv)_x+\Big(hv^2+\dfrac{g}{2}h^2\Big)_y=-ghB_y, \end{cases} \tag{1.1}$$

where t is the time, x and y are horizontal spatial coordinates, $h(x,y,t)$ is the water depth above the bottom, $u(x,y,t)$ and $v(x,y,t)$ are the x- and y-components of the flow velocity, respectively, $q:=hu$ and $p:=hv$ are the corresponding discharges, $B(x,y)$ is a time-independent bottom topography, and g is the constant acceleration due to gravity.

This system is broadly used in many scientific and engineering applications related to modeling of water flows in rivers, lakes, and coastal areas. Development of accurate and robust numerical methods for (1.1) is a challenging task for several reasons. First of all, the system (1.1) is a nonlinear hyperbolic system of balance laws and thus it admit non-smooth solutions with a quite complicated wave structure. Moreover, it is well-known that initially smooth solutions of (1.1) may break down within a finite time.

Another important aspect is related to numerically balancing the flux and source terms in the discharge equations in (1.1): a good numerical scheme should respect this delicate balance. In order to clarify this point, we first note that the steady-state solutions of (1.1) satisfy

$$\begin{cases} q_x+p_y=0, \\ \Big(hu^2+\dfrac{g}{2}h^2\Big)_x+(huv)_y=-ghB_x, \\ (huv)_x+\Big(hv^2+\dfrac{g}{2}h^2\Big)_y=-ghB_y. \end{cases}$$

This system of time-independent PDEs can be rewritten (for smooth solutions) as

$$\begin{cases} q_x+p_y=0, \\ \left[\dfrac{u^2}{2}+g(h+B)\right]_x+vu_y=0, \\ uv_x+\left[\dfrac{v^2}{2}+g(h+B)\right]_y=0. \end{cases} \tag{1.2}$$

Many solutions arising in applications are, in fact, either steady states satisfying (1.2) or their small perturbations. When the computational mesh is coarse, which is typically the case as fine meshes can be computationally unaffordable, truncation errors in the numerical scheme may trigger artificial waves of a magnitude larger than the size of physically relevant waves to be captured. Therefore, a good numerical scheme should be able to