Advances in Applied Mathematics and Mechanics
Adv. Appl. Math. Mech., Vol. **15**, No. 5, pp. 1191-1215

DOI: 10.4208/aamm.OA-2022-0340
October 2023

A Self-Adaptive Algorithm of the Clean Numerical Simulation (CNS) for Chaos

Shijie Qin[2] and Shijun Liao[1,2,3,4,*]

[1] *State Key Laboratory of Ocean Engineering, Shanghai 200240, China*
[2] *Center of Marine Numerical Experiment, School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai 200240, China*
[3] *State Key Laboratory of Plateau Ecology and Agriculture, Xining, Qinghai 810018, China*
[4] *School of Hydraulic and Electric Engineering, Qinghai University, Xining, Qinghai 810018, China*

Received 19 December 2022; Accepted (in revised version) 23 February 2023

Abstract. The background numerical noise ε_0 is determined by the maximum of truncation error and round-off error. For a chaotic system, the numerical error $\varepsilon(t)$ grows exponentially, say, $\varepsilon(t)=\varepsilon_0\exp(\kappa t)$, where $\kappa>0$ is the so-called noise-growing exponent. This is the reason why one can not gain a convergent simulation of chaotic systems in a long enough interval of time by means of traditional algorithms in double precision, since the background numerical noise ε_0 might stop decreasing because of the use of double precision. This restriction can be overcome by means of the clean numerical simulation (CNS), which can decrease the background numerical noise ε_0 to any required tiny level. A lot of successful applications show the novelty and validity of the CNS. In this paper, we further propose some strategies to greatly increase the computational efficiency of the CNS algorithms for chaotic dynamical systems. It is highly suggested to keep a balance between truncation error and round-off error and besides to progressively enlarge the background numerical noise ε_0, since the exponentially increasing numerical noise $\varepsilon(t)$ is much larger than it. Some examples are given to illustrate the validity of our strategies for the CNS.

AMS subject classifications: 65P20, 65Y20
Key words: Chaos, Clean Numerical Simulation (CNS), self-adaptive algorithm, computational efficiency.

1 Introduction

For a chaotic dynamical system, the sensitivity dependence on initial conditions (SDIC) was first discovered by Poincaré [38], and then this phenomenon was discovered once

*Corresponding author.
Email: sjliao@sjtu.edu.cn (S. Liao)

again by Lorenz [32] with a more familiar name "butterfly-effect". Due to the SDIC, a very weak, small-scale disturbance of the initial condition will give rise to a huge deviation of numerical solution of the chaotic system after a long enough temporal interval [9, 16, 45, 50]. Furthermore, it was found that a chaotic dynamical system not only has the sensitivity dependence on initial conditions (SDIC) but also possesses the sensitivity dependence on numerical algorithms (SDNA), as reported by Lorenz [33, 34]. All of these phenomena are due to the exponential increase of noise (or uncertainty) of chaotic systems, but unfortunately *artificial* numerical noises (i.e., truncation errors and round-off errors) are always *inevitable* for almost all of the numerical algorithms. Thus, for a chaotic dynamical system, calculated trajectories of computer-generated simulations obtained by means of different numerical algorithms (with single/double precision) and different time steps are mostly quite different. Naturally, such kind of non-replicability/unreliability of chaotic solution has brought plenty of heated debates on the credence of the numerical simulation of chaotic dynamical system [35], and someone even made an extremely pessimistic conclusion that "for chaotic systems, numerical convergence cannot be guaranteed *forever*" [47]. In addition, it has been recently reported that "shadowing solutions can be almost surely nonphysical", which "invalidates the argument that small perturbations in a chaotic system can only have a small impact on its statistical behavior" [4].

To gain a *reproducible/reliable* numerical simulation of chaotic systems, Liao [23] proposed a brand-new numerical strategy, namely the "Clean Numerical Simulation" (CNS) [24, 25, 27, 30], to control the background numerical noise, say, truncation error and round-off error, during a temporal interval $t \in [0, T_c]$, where T_c is the so-called "critical predictable time" and this temporal interval should be long enough for calculating statistics. In the frame of the CNS [13, 23–25, 27, 29, 30, 39, 41, 53], the temporal truncation error and the spatial truncation error are able to be decreased to a *required* small level via using the Taylor expansion method with a high *enough* order in the temporal dimension and adopting a fine *enough* discretization method in the spatial dimension (such as the high-order spatial Fourier expansion), respectively. Significantly, all of the physical and numerical variables/parameters should be represented by means of the multiple precision (MP) [37] with a large *enough* number of significant digits, and thus the round-off error is also able to be decreased to a *required* small level. Moreover, an additional numerical simulation of the identical chaotic system with the even smaller numerical noise is required and performed in order to determine such a "critical predictable time" T_c, so that the numerical noise could be negligible and thus the computer-generated solution of a chaotic system is reproducible/reliable within the whole spatial computational domain and in the temporal interval $[0, T_c]$. In this way, different from some other general numerical algorithms, the CNS is able to give the reproducible/reliable numerical simulation of a chaotic dynamical system within a finite but long enough temporal interval.

Here it should be emphasized that although our CNS strategy is based on the classical Taylor series method [6] as well as the multiple precision [37], the scientific significance of this strategy is mainly about the "critical predictable time" T_c: the CNS can greatly