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Strong T-Stability of Picard Iteration in a Non-Normal Cone Metric Space

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Abstract. Let (X,d) be a cone metric space and $T : X \to X$ be a mapping. In this paper, we shall introduce the concept of strong T-stability of fixed point iteration procedures with respect to T in cone metric spaces. Also, we will investigate some meaningful results on strong T-stability of Picard iterations in cone metric spaces without the assumption of normality. Our main results improve and generalize some related results in the literature.

Key Words: Strong T-stability, Picard iteration, non-normal cone, cone metric space.

AMS Subject Classifications: 54H25, 47H10

1 Introduction

It is known that the stability theory for fixed point iteration procedures play an important role in the study of fixed point. That is why several scientists paid more attention to the study of the stability (see [1–9]). Since the concept of cone metric spaces, as a generalization of metric spaces, was mentioned by Huang and Zhang [10], many scholars have studied the stability for iteration procedures in cone metric spaces (see [1,2]). However, they got the stability of Picard iteration only in the case of normal cone. Rezapour and Hamlbarani [11], making the best of solid cone, provided an example of non-normal cone and omitted the assumption of normality in some results of Huang and Zhang. In this work, we will introduce the concept of strong T-stability of fixed point iteration procedures with respect to T and study some results on strong T-stability of Picard iteration in cone metric spaces without the assumption of normality, where $T : X \to X$ is a

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self-mapping defined on a cone metric space (X,d). Moreover, we find some interesting and important equivalent conclusions on strong T-stability of Picard iterations. Obviously, our results generalize, extend and unify several well known comparable results (see [1–5]).

Consistent with Huang and Zhang [10], the following definitions and results will be needed in the sequel.

Let E be a real Banach space and P a subset of E. By θ we denote the zero element of E and by int$P$ the interior of P. The subset P is called a cone if and only if:

(i) P is closed, nonempty, and $P \neq \{\theta\}$;

(ii) $a,\ b \in \mathbb{R}, a,b \geq 0, x,y \in P \Rightarrow ax+by \in P$;

(iii) $P \cap (-P) = \{\theta\}$.

On this basis, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y-x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y-x \in \text{int}P$, where int$P$ denotes the interior of the cone P. Write $\|\cdot\|$ as the norm on E. The cone P is called normal if there is a number $K>0$ such that for all $x,y \in E$, $\theta \preceq x \preceq y$ implies $\|x\| \leq K\|y\|$. The least positive number satisfying above is called the normal constant of P. It is well known that $K \geq 1$.

In the following we always suppose that E is a real Banach space, P is a cone in E with int$P \neq \emptyset$ and $\preceq$ is a partial ordering with respect to P.

Definition 1.1 ([10]). *Let X be a nonempty set. Suppose that the mapping $d: X \times X \to E$ satisfies:*

(d1) $\theta \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y)=\theta$ if and only if $x=y$;

(d2) $d(x,y)=d(y,x)$ for all $x,y \in X$;

(d3) $d(x,y) \preceq d(x,z)+d(z,y)$ for all $x,y,z \in X$.

Then d is called a cone metric on X and (X,d) is called a cone metric space.

The class of cone metric spaces is more general than that of metric spaces, because each metric space is a cone metric space where $E=\mathbb{R}$ and $P=[0,+\infty)$.

Definition 1.2 ([10]). *Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n,x) \ll c$ for all $n \geq N$. We denote this by

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x, \quad (n \to \infty).$$

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n,x_m) \ll c$ for all $n,m \geq N$.

(iii) (X,d) is a complete cone metric space if every Cauchy sequence is convergent.