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Existence of Weak Solution for $p(x)$-Kirchhoff Type Problem Involving the $p(x)$-Laplacian-like Operator by Topological Degree

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Abstract. In this paper, we study the existence of "weak solution" for a class of $p(x)$-Kirchhoff type problem involving the $p(x)$-Laplacian-like operator depending on two real parameters with Neumann boundary condition. Using a topological degree for a class of demicontinuous operator of generalized (S_+) type and the theory of the variable exponent Sobolev space, we establish the existence of "weak solution" of this problem.

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1 Introduction

The study of differential equations and variational problems with nonlinearities and nonstandard $p(x)$-growth conditions or nonstandard $(p(x),q(x))-$ growth conditions have received a lot of attention. Perhaps the impulse for this comes from the new search field that reflects a new type of physical phenomenon is a class of nonlinear problems with variable exponents (see [1–3]). The motivation for this research comes from the application of similar models in physics to represent the behavior of elasticity [4] and electrorheological fluids (see [5, 6]), which have the ability to modify their mechanical properties

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when exposed to an electric field (see [7–10]), specifically the phenomenon of capillarity, which depends on solid-liquid interfacial characteristics as surface tension, contact angle, and solid surface geometry.

Let Ω be a bounded domain in $\mathbb{R}^N (N>1)$ with smooth boundary denoted by $\partial\Omega$, $R\in L^\infty(\Omega)$, $p(x)$, $c(x)\in C_+(\overline{\Omega})$(that will be defined in the Preliminaries), and let μ and λ be two real parameters.

In this paper, we establish the existence of weak solution for a class of $p(x)$-Kirchhoff type problem involving the $p(x)$-Laplacian-like operator depending on two real parameters with Neumann boundary condition of the following form:

$$\begin{cases} -\mathcal{M}\Big(\Theta(u)\Big)\Big(\Delta^{\mathcal{L}}_{p(x)}u-|u|^{p(x)-2}u\Big)+R(x)|u|^{c(x)-2}u \\ \qquad =\mu g(x,u)+\lambda f(x,u,\nabla u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \eta}=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where

$$\Theta(u):=\int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)}+\sqrt{1+|\nabla u|^{2p(x)}}+|u|^{p(x)}\Big)\mathrm{d}x,$$

and

$$\Delta^{\mathcal{L}}_{p(x)}u:=\mathrm{div}\Big(|\nabla u|^{p(x)-2}\nabla u+\frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)$$

is the $p(x)$-Laplacian-like operator, $g:\Omega\times\mathbb{R}\to\mathbb{R}$ and $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ are Carathéodory functions that satisfy the assumption of growth and $\mathcal{M}:\mathbb{R}^+\to\mathbb{R}^+$ is a continuous function.

Problems related to (1.1) have been studied by many scholars, for example, Ni and Serrin [11, 12] considered the following equation

$$-\mathrm{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)=f(u) \quad \text{in } \mathbb{R}^N. \tag{1.2}$$

The operator on the left-hand side of (1.2) is most often denoted by the specified mean curvature operator and $\nabla u/\sqrt{1+|\nabla u|^2}$ is the Kirchhoff stress term.

In the case when $\mathcal{M}\Big(\Theta(u)\Big)\equiv 1$, $\mu=R=0$, $\lambda>0$, f independent of ∇u and without the term $|u|^{p(x)-2}u$ with Dirichlet boundary condition, we know that the problem (1.1) has a nontrivial solution from [13] (see also [14–17]).

For $\mathcal{M}\Big(\Theta(u)\Big)\equiv 1$, $c(x)=p(x)$, $\mu\geq 0$, $\lambda>0$, $R\in L^\infty(\Omega)$ with $\mathrm{essinf}_\Omega R>0$ and f independent of ∇u, Afrouzi et al. [18] established some new sufficient conditions underwhich the problem (1.1) possesses infinitely many weak solutions. Their discussion is based on a fully variational method and the main tool is a general critical point theorem.