Commun. Math. Res.
doi: 10.4208/cmr.2023-0003

Vol. **39**, No. 4, pp. 501-522
November 2023

Approximation of the Long-Time Dynamics of the Dynamical System Generated by the Ginzburg-Landau Equation

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Received 12 January 2023; Accepted 24 April 2023

Abstract. In this article we consider the (complex) Ginzburg-Landau equation, we discretize in time using the implicit Euler scheme, and with the aid of the discrete Gronwall lemma and of the discrete uniform Gronwall lemma we prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the time-step approaches zero.

AMS subject classifications: 35Q56, 35Q35

Key words: Ginzburg-Landau equation, implicit Euler scheme, long-time stability, attractors.

1 Introduction

In this article, we consider a nonlinear Schrödinger equation with a nonlinear term, the Ginzburg-Landau equation. Ginzburg-Landau equations have been used to model a wide variety of physical systems such as the development of Tollmieu-Schlichting waves in plane Poiseuille flows, the nonlinear growth of convection rolls in the Rayleigh-Bénard problem, the appearance of Taylor vor-

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tices in the flow between counterrotating circular cylinders (see, e.g., [1, 6–8, 10]). The equation also arises in the study of chemical systems governed by reaction-diffusion equations.

In this work, we discretize the Ginzburg-Landau equation in time using the implicit Euler scheme and with the aid of the discrete Gronwall lemma and of the discrete uniform Gronwall lemma we prove that the global attractors generated by the numerical scheme converge to the global attractor of the continuous system as the timestep approaches zero. Our work has been inspired by previous results of one of the authors and her collaborators. In [14], for example, the authors considered the implicit Euler scheme for the 2D Navier-Stokes equations and proved that the numerical scheme was H^1-uniformly stable in time. In a later article (see [2]), the authors used the theory for multi-valued attractors to prove the convergence of the discrete attractors to the global attractor of the continuous system as the time-step parameter approached zero. In [12], the author considered the implicit Euler scheme for the two-dimensional magnetohydrodynamics equations and showed that the scheme was H^2-stable. Similar results were obtained in [13] and [3], where the authors proved not only the long-time stability of the implicit Euler scheme for the two-dimensional Rayleigh-Benard convection problem, and the thermohydraulics equations, respectively, but also the convergence of the global attractors generated by the numerical scheme to the global attractor of the continuous system as the time-step approaches zero.

This article is divided as follows. In the next section, we recall from [11] the Ginzburg-Landau equation and its mathematical setting. In Section 3, we study the stability of a time discretization scheme for the model. More precisely, we prove that the scheme is uniformly bounded in the spaces $\mathbb{L}^2$ and $\mathbb{H}_0^1$. In Section 4, we prove the uniqueness of the discrete solution, provided that the time-step is small enough. The dependance of the time-step on the initial data prevents us from defining a single-valued attractor in the classical sense. This is why in Section 5, we recall the theory of the so-called multi-valued attractors, and then we apply it to our model.

2 The Ginzburg-Landau equation and its mathematical setting

Let $\Omega \subset \mathbb{R}^n$, $n=1,2$, be an open bounded set with boundary $\partial\Omega$ of class C^2. The Ginzburg-Landau equation is given by

$$u_t-(\lambda+i\alpha)\Delta u+(\kappa+i\beta)|u|^2u-\gamma u=0, \tag{2.1}$$