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# AN INDEFINITE-PROXIMAL-BASED STRICTLY CONTRACTIVE PEACEMAN-RACHFORD SPLITTING METHOD[*1)]

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**Abstract**

The Peaceman-Rachford splitting method is efficient for minimizing a convex optimization problem with a separable objective function and linear constraints. However, its convergence was not guaranteed without extra requirements. He *et al.* (SIAM J. Optim. 24: 1011 - 1040, 2014) proved the convergence of a strictly contractive Peaceman-Rachford splitting method by employing a suitable underdetermined relaxation factor. In this paper, we further extend the so-called strictly contractive Peaceman-Rachford splitting method by using two different relaxation factors. Besides, motivated by the recent advances on the ADMM type method with indefinite proximal terms, we employ the indefinite proximal term in the strictly contractive Peaceman-Rachford splitting method. We show that the proposed indefinite-proximal strictly contractive Peaceman-Rachford splitting method is convergent and also prove the $o(1/t)$ convergence rate in the nonergodic sense. The numerical tests on the $l_1$ regularized least square problem demonstrate the efficiency of the proposed method.

*Mathematics subject classification:* 90C25, 90C30, 65K05.
*Key words:* Indefinite proximal, Strictly contractive, Peaceman-Rachford splitting method, Convex minimization, Convergence rate.

## 1. Introduction

We consider the convex minimization problem with linear constraints and a separable objective function:

$$\min \theta_1(x) + \theta_2(y), \quad \text{s.t.} \quad Ax + By = b, \quad x \in \mathcal{X}, \quad y \in \mathcal{Y}, \tag{1.1}$$

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where $A \in \mathbb{R}^{m\times n_1}$ and $B \in \mathbb{R}^{m\times n_2}$, $b \in \mathbb{R}^m$, $\mathcal{X} := \mathbb{R}^{n_1}$, $\mathcal{Y} := \mathbb{R}^{n_2}$. The functions $\theta_1(x) := p(x)+f(x)$ and $\theta_2(y) := h(y) + g(y)$, where $p : \mathcal{X} \to (-\infty, +\infty]$ and $h : \mathcal{Y} \to (-\infty, +\infty]$ are proper closed convex (could be nonsmooth) functions; $f : \mathcal{X} \to (-\infty, +\infty)$ and $g : \mathcal{Y} \to (-\infty, +\infty)$ are two convex functions with Lipschitz continuous gradients on $\mathcal{X}$ and $\mathcal{Y}$. Throughout, the solution set of (1.1) is assumed to be nonempty. Note that one can also consider $\mathcal{X}$ and $\mathcal{Y}$ as general real finite dimensional Euclidean or Hilbert spaces, see [3,15,35] for instances. For ease of presentation, we adopt the $\mathcal{X}$ and $\mathcal{Y}$ as the ordinary $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ in this paper.

Let $\mathcal{L}_\beta(x, y, \lambda)$ be the augmented Lagrangian function for (1.1) that defined by

$$\mathcal{L}_\beta(x, y, \lambda) := \theta_1(x) + \theta_2(y) - \langle \lambda, Ax + By - b\rangle + \frac{\beta}{2}\|Ax + By - b\|^2, \tag{1.2}$$

in which $\lambda \in \mathbb{R}^m$ is the multiplier associated to the linear constraint and $\beta > 0$ is a penalty parameter.

A well-known method called alternating direction method of multipliers (ADMM) is efficient to minimize such problems. It was observed in Gabay and Mercier [11], Glowinski and Marrocco [13] that ADMM can be derived from applying the Douglas-Rachford operator splitting method [8] to the dual of the problem (1.1). The iterative sequence is given as the following recursion:

$$\begin{cases} x^{k+1} = \arg\min_{x\in\mathcal{X}} \mathcal{L}_\beta(x, y^k, \lambda^k), & (1.3\text{a})\\ y^{k+1} = \arg\min_{y\in\mathcal{Y}} \mathcal{L}_\beta(x^{k+1}, y, \lambda^k), & (1.3\text{b})\\ \lambda^{k+1} = \lambda^k - \beta(Ax^{k+1} + By^{k+1} - b). & (1.3\text{c}) \end{cases}$$

Based on another classical operator splitting method, i.e., the Peaceman-Rachford operator splitting method [30], one can derive the following similar method for (1.1):

$$\begin{cases} x^{k+1} = \arg\min_{x\in\mathcal{X}} \mathcal{L}_\beta(x, y^k, \lambda^k), & (1.4\text{a})\\ \lambda^{k+\frac{1}{2}} = \lambda^k - \beta(Ax^{k+1} + By^k - b), & (1.4\text{b})\\ y^{k+1} = \arg\min_{y\in\mathcal{Y}} \mathcal{L}_\beta(x^{k+1}, y, \lambda^{k+\frac{1}{2}}), & (1.4\text{c})\\ \lambda^{k+1} = \lambda^{k+\frac{1}{2}} - \beta(Ax^{k+1} + By^{k+1} - b). & (1.4\text{d}) \end{cases}$$

While the global convergence of the alternating direction method of multipliers (1.3a)-(1.3c) can be established under very mild conditions [2], the convergence of the Peaceman-Rachford-based method (1.4a)-(1.4d) can not be guaranteed without further conditions [5].

He *et al.* [17] proposed a modification of (1.4a)-(1.4d) by introducing a parameter $\alpha$ to the update scheme of the dual variable $\lambda$ in (1.4b) and (1.4d). Note that when $\alpha = 1$, it is the same as (1.4a)-(1.4d). They explained the non-convergence behavior of (1.4a)-(1.4d) from the contractive perspective, i.e., the distance from the iterative point to the solution set is merely nonexpansive, but not contractive. Under the condition that $\alpha \in (0, 1)$, they proved the same sublinear convergence rate as that for ADMM [20]. Particularly, they showed that it achieves an approximate solution of (1.1) with the accuracy of $\mathcal{O}(1/t)$ after $t$ iterations[1)] , both in the ergodic and nonergodic sense. Besides, Gu [14] and He *et al.* [18] took two different constants $\alpha$ and $\gamma$ to different step sizes in (1.4b) and (1.4d). The convergence results, including global

[1)] A worst-case $\mathcal{O}(1/t)$ convergence rate means the accuracy to a solution under certain criteria is of the order $\mathcal{O}(1/t)$ after $t$ iterations of an iterative scheme; or equivalently, it requires at most $\mathcal{O}(1/\epsilon)$ iterations to achieve an approximate solution with an accuracy of $\epsilon$. See, e.g., [27,28].