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An Extended Two-Step Method for Inverse Eigenvalue Problems with Multiple Eigenvalues

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Abstract. In recent years, numerical solutions of the inverse eigenvalue problems with multiple eigenvalues have attracted the attention of some researchers, and there have been a few algorithms with quadratic convergence. We propose here an extended two-step method for solving the inverse eigenvalue problems with multiple eigenvalues. Under appropriate assumptions, the convergence analysis of the extended method is presented and the cubic root-convergence rate is proved. Numerical experiments are provided to confirm the theoretical results and comparisons with the inexact Cayley transform method are made. Our extended method and convergence result in the present paper may enrich the results of numerical solutions of the inverse eigenvalue problems with multiple eigenvalues.

AMS subject classifications: 65F18, 65F10, 15A18

Key words: Inverse eigenvalue problems, extended two-step method, cubic root-convergence.

1. Introduction

The applications of inverse eigenvalue problems (IEPs) are fully extensive, such as the inverse Strum-Liouville problems, the inverse Toeplitz eigenvalue problems, the inverse vibrating string problem, applied mechanics and structure design, nuclear spectroscopy, molecular spectroscopy and so on. For different applications, mathematical theories and numerical methods about IEPs, one may refer to [1, 3, 7, 9, 11, 12, 14, 15, 18, 20, 27, 29].

In recent years, many mathematicians have carried out in-depth research on the theoretical analysis and algorithm implementation of the following inverse eigenvalue problem (IEP). Let $\{A_i\}_{i=0}^n$ be $(n+1)$ real symmetric n-by-n matrices. We define the operator A from $\mathbb{R}^n$ to $\mathbb{R}^{n\times n}$ by

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$$A(\mathbf{c}) := A_0 + \sum_{i=1}^{n} c_i A_i, \quad \mathbf{c} = (c_1, c_2, \ldots, c_n)^T \in \mathbb{R}^n. \tag{1.1}$$

For any $\mathbf{c} \in \mathbb{R}^n$, let $\{\lambda_i(\mathbf{c})\}_{i=1}^n$ be the eigenvalues of $A(\mathbf{c})$ satisfying $\lambda_1(\mathbf{c}) \leq \lambda_2(\mathbf{c}) \leq \cdots \leq \lambda_n(\mathbf{c})$. The IEP studied in this paper is, for given n real numbers $\{\lambda_i^*\}_{i=1}^n$ with the order

$$\lambda_1^* \leq \lambda_2^* \leq \cdots \leq \lambda_n^*,$$

to find a vector $\mathbf{c}^* \in \mathbb{R}^n$ such that

$$\lambda_i(\mathbf{c}^*) = \lambda_i^*, \quad i = 1, 2, \ldots, n. \tag{1.2}$$

Then, the vector $\mathbf{c}^*$ is called a solution of the IEP (1.2).

As mentioned in [10], there is a large literature on conditions for existence and uniqueness of solutions to the IEP (1.2) in many special cases. In this paper, we consider the formulation and local analysis of the numerical algorithm for solving the IEP (1.2). As described in [4, 5, 22], solving the IEP (1.2) is equivalent to solving the following nonlinear equation:

$$\mathbf{f}(\mathbf{c}) := \big(\lambda_1(\mathbf{c}) - \lambda_1^*,\ \lambda_2(\mathbf{c}) - \lambda_2^*, \ldots, \lambda_n(\mathbf{c}) - \lambda_n^*\big)^T = \mathbf{0}, \quad \mathbf{c} \in \mathbb{R}^n. \tag{1.3}$$

When the given eigenvalues are distinct, i.e.,

$$\lambda_1^* < \lambda_2^* < \cdots < \lambda_n^*,$$

there exists a neighborhood of $\mathbf{c}^*$ where the function $\mathbf{f}$ is analytic (cf. [4]). Thus, based on the equivalence of the IEP (1.2) and the nonlinear equation (1.3), Newton's method and various Newton-type methods were proposed to solve the IEP (1.2) in the distinct case [2,4,10,21,22]. However, when multiple eigenvalues are present, solving the IEP (1.2) becomes more complicated in the sense that the eigenvalues are not in general differentiable at $\mathbf{c}^*$ and that the eigenvectors of $A(\mathbf{c}^*)$ are not unique, and they cannot generally be defined to be continuous functions of $\mathbf{c}$ at $\mathbf{c}^*$. Nevertheless, some numerical methods for solving the IEP (1.2) were extended to the multiple case and the convergence theorems of the extended methods were presented, see for instance [23–25].

Note that all the numerical methods in the mentioned references above are quadratic convergent. In recent years, there have been some algorithms with super quadratic convergence rate or cubic root-convergence rate. Particularly, Wen *et al.* [6] proposed a super quadratic convergent two-step Newton-type method where the approximate Jacobian equations are solved by inexact methods. In 2020, they improved the work [6] and designed a two-step inexact Newton-Chebyshev-like method with cubic root-convergence rate which avoids solving the approximate Jacobian equations by using the Chebyshev method (cf. [28]). While Ma [16] introduced a two-step Ulm-Chebyshev-like Cayley transform method with cubic root-convergence rate which also