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Neural Networks with Local Converging Inputs (NNLCI) for Solving Conservation Laws, Part II: 2D Problems

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Abstract. In our prior work [10], neural networks with local converging inputs (NNLCI) were introduced for solving one-dimensional conservation equations. Two solutions of a conservation law in a converging sequence, computed from low-cost numerical schemes, and in a local domain of dependence of the space-time location, were used as the input to a neural network in order to predict a high-fidelity solution at a given space-time location. In the present work, we extend the method to two-dimensional conservation systems and introduce different solution techniques. Numerical results demonstrate the validity and effectiveness of the NNLCI method for application to multi-dimensional problems. In spite of low-cost smeared input data, the NNLCI method is capable of accurately predicting shocks, contact discontinuities, and the smooth region of the entire field. The NNLCI method is relatively easy to train because of the use of local solvers. The computing time saving is between one and two orders of magnitude compared with the corresponding high-fidelity schemes for two-dimensional Riemann problems. The relative efficiency of the NNLCI method is expected to be substantially greater for problems with higher spatial dimensions or smooth solutions.

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Key words: Neural network, neural networks with local converging inputs, physics informed machine learning, conservation laws, differential equation, multi-fidelity optimization.

1 Introduction

Artificial neural networks [9] are an important tool for computations in science and engineering. Many approaches have recently been developed that incorporate artificial

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neural networks for solving partial differential equations. For example, Sirignano and Spiliopoulos [29] introduced the Deep Galerkin Method to approximate the unknown solution as a mapping from a space-time location to the solution value there with a deep neural network, incorporating the finite difference residue error and initial and boundary constraints in the loss function. E and Yu [7] introduced the Deep Ritz Method, which incorporates the Ritz energy of a finite element method into the loss function. Raissi *et al.* [24] developed physics-informed neural networks (PINN) by employing an automatic differentiation [3] to define the residue error in the loss function. Much success has been achieved in predicting a variety of flow problems with given governing equations, including the Navier-Stokes system [23–26], hypersonic flow [18], electro-convection [21] and others. In [20], the Rankine-Hugoniot jump conditions were added as a constraint to the loss function of the neural network for solving the Riemann problems. In [14], a specially designed neural network was used to approximate the mapping from all known information, such as initial and boundary values, to the unknown solution, and many existing solutions have been used to train such a neural network. In [5, 19], finite expansions of neural networks that can be trained off-line were introduced to form a mapping from the initial value and a spatial location to a later high-fidelity solution at the same location.

Neural networks have also been trained off-line to predict key parameters of a numerical scheme. In [2, 6, 27], neural networks were used to detect discontinuities. An appropriate slope limiter or artificial viscosity was then determined to treat discontinuities using a local solution as the input. Another approach is to use a low-cost numerical solution computed on a coarse grid as input to predict a high-fidelity solution [15, 22].

In our earlier work [10], a novel neural network method (NNLCI) was introduced to solve conservation laws whose solutions may contain shock and contact discontinuities. In NNLCI, local low-cost solutions are employed as the input to a neural network to predict a high-fidelity solution at a given space-time location. To enable the neural network to distinguish a numerically smeared discontinuity from a smooth solution with large gradient in its input, the input is created by solving the conservation laws twice in sequence, with approximate solutions of converging accuracy, with low-cost numerical schemes and in a local domain of dependence of the space-time location. Because a numerical discontinuity becomes increasingly steeper in a converging sequence in the input, while a smooth solution does not, the neural network then can accurately identify flow attributes in its input and make the correct prediction. Such inputs can be generated in different ways, including schemes with two different grids (with one grid coarser than the other), with two different numerical diffusion coefficients on the same grid, or with two schemes of different orders of accuracy on the same grid. All inputs and high-fidelity solutions for all cases studied throughout the paper are computed by a first-order or fourth-order numerical scheme, using Dual Intel Xeon Gold 6226 processors. The NNLCI approach works effectively, not only for discontinuities, but also for smooth regions of the solution. It has broad application to a wide variety of differential equations. The computational cost is modest because it is a local post-processing-type solver, and