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# Genuinely Multidimensional Physical-Constraints-Preserving Finite Volume Schemes for the Special Relativistic Hydrodynamics

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**Abstract.** This paper develops the genuinely multidimensional HLL Riemann solver for the two-dimensional special relativistic hydrodynamic equations on Cartesian meshes and studies its physical-constraint-preserving (PCP) property. Based on the resulting HLL solver, the first- and high-order accurate PCP finite volume schemes are proposed. In the high-order scheme, the WENO reconstruction, the third-order accurate strong-stability-preserving time discretizations and the PCP flux limiter are used. Several numerical results are given to demonstrate the accuracy, performance and resolution of the shock waves and the genuinely multi-dimensional wave structures etc. of our PCP finite volume schemes.

**AMS subject classifications**: 65M08, 35L02, 76Y05, 83A05

**Key words**: Genuinely multidimensional schemes, HLL, physical-constraint-preserving property, high order accuracy, special relativistic hydrodynamics.

## 1 Introduction

The paper is concerned with the genuinely multidimensional physical-constraints-preserving (PCP) finite volume schemes for the special relativistic hydrodynamics (RHD), which plays a crucial role in astrophysics, plasma physics, and nuclear physics etc. This is particularly relevant in scenarios where fluids move at extremely high velocities close to the speed of light, necessitating the consideration of relativistic effects. In

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the (rest) laboratory frame, the two-dimensional (2D) special RHD equations governing an ideal fluid flow can be expressed in the divergence form

$$\frac{\partial \boldsymbol{U}}{\partial t}+\sum_{\ell=1}^{2}\frac{\partial \boldsymbol{F}_\ell(\boldsymbol{U})}{\partial x_\ell}=0, \tag{1.1}$$

where the conservative vector $\boldsymbol{U}$ and the flux $\boldsymbol{F}_\ell$ are defined respectively by

$$\boldsymbol{U}=(D,\boldsymbol{m},E)^T,\quad \boldsymbol{F}_\ell=\left(Du_\ell,\boldsymbol{m}u_\ell+p\boldsymbol{e}_\ell,(E+p)u_\ell\right)^T,\quad \ell=1,2. \tag{1.2}$$

Here $D=\rho\gamma$, $\boldsymbol{m}=Dh\gamma\boldsymbol{u}$ and $E=Dh\gamma-p$ are the mass, momentum and total energy relative to the laboratory frame respectively, $p$ denotes the gas pressure, $\boldsymbol{u}=(u_1,u_2)$ is the fluid velocity vector, $\boldsymbol{e}_\ell$ is the row vector denoting the $\ell$-th row of the unit matrix of size 2, $\rho$ is the rest-mass density, $\gamma=1/\sqrt{1-|\boldsymbol{u}|^2}$ is the Lorentz factor, $|\boldsymbol{u}|^2=u_1^2+u_2^2$, $h=1+e+\frac{p}{\rho}$ is the specific enthalpy, and $e$ is the specific internal energy. Note that natural unit (i.e., the speed of light $c=1$) has been used. The closure of the system (1.1) should be accomplished by incorporating the equation of state (EOS), which has a general form of $p=p(\rho,e)$. For simplicity, this paper considers the EOS for the perfect gas, namely

$$p=(\Gamma-1)\rho e, \tag{1.3}$$

with the adiabatic index $\Gamma\in(1,2]$. Such restriction on $\Gamma$ is reasonable under the compressibility assumptions, and $\Gamma$ is taken as 5/3 for the mildly relativistic case and 4/3 for the ultra-relativistic case. In this case, for $i=1,2$, the Jacobian matrix $\boldsymbol{A}_i(\boldsymbol{U})=\partial\boldsymbol{F}_i/\partial\boldsymbol{U}$ of the system (1.1) has 4 real eigenvalues, listed in ascending order as follows

$$\begin{aligned}
\lambda_i^{(1)}(\boldsymbol{U})&=\frac{u_i(1-c_s^2)-c_s\gamma^{-1}\sqrt{1-u_i^2-c_s^2(|\boldsymbol{u}|^2-u_i^2)}}{1-c_s^2|\boldsymbol{u}|^2},\\
\lambda_i^{(2)}(\boldsymbol{U})&=\lambda_i^{(3)}(\boldsymbol{U})=u_i,\\
\lambda_i^{(4)}(\boldsymbol{U})&=\frac{u_i(1-c_s^2)+c_s\gamma^{-1}\sqrt{1-u_i^2-c_s^2(|\boldsymbol{u}|^2-u_i^2)}}{1-c_s^2|\boldsymbol{u}|^2},
\end{aligned}$$

where $c_s$ is the speed of sound expressed explicitly by

$$c_s=\sqrt{\Gamma p/(\rho h)},$$

and satisfies

$$c_s^2=\frac{\Gamma p}{\rho h}=\frac{\Gamma p}{\rho+\frac{p}{\Gamma-1}+p}=\frac{(\Gamma-1)\Gamma p}{(\Gamma-1)\rho+\Gamma p}<\Gamma-1\leq 1=c.$$

Due to the relativistic effects, especially the appearance of the Lorentz factor, the system (1.1) exhibits much stronger nonlinearity compared to the non-relativistic case,