Commun. Comput. Phys.
doi: 10.4208/cicp.OA-2023-0075

Vol. **34**, No. 4, pp. 1133-1176
October 2023

Bi-Orthogonal fPINN: A Physics-Informed Neural Network Method for Solving Time-Dependent Stochastic Fractional PDEs

Lei Ma[1], Rongxin Li[1], Fanhai Zeng[2,*], Ling Guo[1,*] and George Em Karniadakis[3]

[1] *Department of Mathematics, Shanghai Normal University, Shanghai 200234, China.*
[2] *School of Mathematics, Shandong University, Shandong 250100, China.*
[3] *Division of Applied Mathematics, Brown University, Providence, RI 02906, USA, and School of Engineering, Brown University, Providence, RI 02906, USA.*

Received 11 March 2023; Accepted (in revised version) 4 August 2023

Abstract. Fractional partial differential equations (FPDEs) can effectively represent anomalous transport and nonlocal interactions. However, inherent uncertainties arise naturally in real applications due to random forcing or unknown material properties. Mathematical models considering nonlocal interactions with uncertainty quantification can be formulated as stochastic fractional partial differential equations (SFPDEs). There are many challenges in solving SFPDEs numerically, especially for long-time integration since such problems are high-dimensional and nonlocal. Here, we combine the bi-orthogonal (BO) method for representing stochastic processes with physics-informed neural networks (PINNs) for solving partial differential equations to formulate the bi-orthogonal PINN method (BO-fPINN) for solving time-dependent SFPDEs. Specifically, we introduce a deep neural network for the stochastic solution of the time-dependent SFPDEs, and include the BO constraints in the loss function following a weak formulation. Since automatic differentiation is not currently applicable to fractional derivatives, we employ discretization on a grid to compute the fractional derivatives of the neural network output. The weak formulation loss function of the BO-fPINN method can overcome some drawbacks of the BO methods and thus can be used to solve SFPDEs with eigenvalue crossings. Moreover, the BO-fPINN method can be used for inverse SFPDEs with the same framework and same computational complexity as for forward problems. We demonstrate the effectiveness of the BO-fPINN method for different benchmark problems. Specifically, we first consider an SFPDE with eigenvalue crossing and obtain good results while the original BO method fails. We then solve several forward and inverse problems governed by SFPDEs, including problems with noisy initial conditions. We study the effect of the fractional order as

*Corresponding author. *Email addresses:* `1000494866@smail.shnu.edu.cn` (L. Ma), `1000478817@smail.shnu.edu.cn` (R. Li), `fanhai_zeng@sdu.edu.cn` (F. Zeng), `lguo@shnu.edu.cn` (L. Guo), `george_karniadakis@Brown.edu` (G. Karniadakis)

well as the number of the BO modes on the accuracy of the BO-fPINN method. The results demonstrate the flexibility and efficiency of the proposed method, especially for inverse problems. We also present a simple example of transfer learning (for the fractional order) that can help in accelerating the training of BO-fPINN for SFPDEs. Taken together, the simulation results show that the BO-fPINN method can be employed to effectively solve time-dependent SFPDEs and may provide a reliable computational strategy for real applications exhibiting anomalous transport.

AMS subject classifications: 26A33, 65M06, 65M12, 65M15, 35R11

Key words: Scientific machine learning, uncertainty quantification, stochastic fractional differential equations, PINNs, inverse problems.

---

1 Introduction

Fractional partial differential equations (FPDEs) can effectively represent anomalous transport and nonlocal interactions in engineering and medical fields, e.g., porous media, viscoelasticity, non-Newtonian fluid mechanics, soft tissue mechanics, etc. [1–5]. However, simulating real-world applications requires modelers to consider many uncertain factors, such as material properties, random forcing terms, experimental measurement errors, and the complexity of geometric regions with random roughness. These uncertain factors may have an important impact on the system evolution, especially in long-term forecasting, and hence quantifying uncertainty is very important. Following a probability framework, the uncertainty is usually modeled as a random field [6], and therefore modeling nonlocal interactions with uncertainty requires the formulation of fractional partial differential equation with random inputs. Although there have been some achievements in the numerical solution of stochastic fractional partial differential equations (SFPDEs) [7–9], the design of reliable algorithms that can tackle high-dimensions and long-term integration is still an open challenge in the context of solving efficiently time-dependent stochastic fractional partial differential equations (SFPDEs).

Monte Carlo (MC) and Quasi Monte Carlo (QMC) methods are common methods for solving differential equations with random inputs, with the statistical values (e.g., mean and variance) of the random solutions obtained by numerically solving a set of corresponding deterministic differential equations at sample points. However, such methods have slow convergence speed and tax computational resources heavily for large complex systems [10, 11]. In recent years, Polynomial Chaos (PC) has been widely used to solve partial differential equations with random inputs, which can be regarded as a spectral approximation method in random space. The basic idea is to first employ a Karhunen Loève expansion of the random field to reduce the dimension of the infinite-dimensional random inputs and represent them with a finite-dimensional series, and subsequently construct a polynomial surrogate model of the random solution. The polynomial expansion coefficients are solved by the stochastic Galerkin method or the stochastic collocation