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Divergence-Free Virtual Element Method for the Stokes Equations with Damping on Polygonal Meshes

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Abstract. In this paper, we construct, analyze, and numerically validate a family of divergence-free virtual elements for Stokes equations with nonlinear damping on polygonal meshes. The virtual element method is $\mathbf{H}^1$-conforming and exact divergence-free. By virtue of these properties and the topological degree argument, we rigorously prove the well-posedness of the proposed discrete scheme. The convergence analysis is carried out, which imply that the error estimate for the velocity in energy norm does not explicitly depend on the pressure. Numerical experiments on various polygonal meshes validate the accuracy of the theoretical analysis and the asymptotic pressure robustness of the proposed scheme.

AMS subject classifications: 65N15, 35R05, 65N30

Key words: Optimal error estimates, divergence-free, virtual element, nonlinear damping term.

1. Introduction

In the past years, there has been a growing emphasis on studying the Stokes equations with damping due to their widespread applications in fluid mechanics, geophysics, and ocean acoustics [2, 31, 39]. In this paper, we investigate the steady Stokes equations with damping in a polygonal domain $\Omega \subset \mathbb{R}^2$ subject to homogeneous Dirichlet boundary conditions: Find a pair $(\mathbf{u}, p)$ such that

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$$\begin{cases} -\nu\Delta\mathbf{u} + \alpha|\mathbf{u}|^{r-2}\mathbf{u} + \nabla p = \mathbf{f} & \text{in } \Omega, \\ \text{div } \mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\mathbf{u} = (u_1, u_2)$, p and $\mathbf{f}$ represent the fluid velocity, pressure and external force, respectively, $\nu > 0$ is the viscosity coefficient, $2 < r < \infty$ and $\alpha > 0$ are two damping parameters, and $|\cdot|$ denotes the Euclidean norm, i.e. $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$. The damping term arises due to its hindering effect on fluid motion and plays a crucial role in characterizing various physical properties of the fluid, such as flow within porous media, resistance near the Earth's surface in atmospheric flows, friction effects, and other dissipative mechanisms [6, 16, 31, 32].

Over the past few decades, numerous academics have investigated and developed theoretical analyses of partial differential equations (PDEs) with damping terms. For nonlinear hyperbolic problems, the presence of the damping term $\alpha|\mathbf{u}|^{r-2}\mathbf{u}$ may result in blow-up of solution within a finite time [31, 48]. Therefore, it is imperative to conduct rigorous theoretical analyses and extensive simulation experiments to ascertain the critical value and parameter range of the damping term. For fluid problems, Constantin and Ramos [16] conducted an investigation into the long-time behavior of solutions to the two-dimensional Navier-Stokes equations with a linear damping term $\alpha\mathbf{u}$ as the viscosity coefficient approaches zero. Cai *et al.* [8, 9] delved into the long-time behavior of solutions to the Navier-Stokes equations with nonlinear damping terms, and examining the existence of global weak solutions and strong solutions.

On the numerical level, several methods have been developed for solving the Stokes equations with damping term (1.1). Liu and Li [38] proposed and analyzed the mixed finite element method for the Eqs. (1.1), and proved the well-posedness and error estimates of the proposed scheme. In [47], the weak Galerkin method combined with the two-level method were considered for the Stokes equations with damping term on general meshes and optimal error estimates of velocity and pressure were obtained. Then, by using the interior penalty discontinuous Galerkin (IPDG) method, Zhang *et al.* [57] proved the consistency and stabilization of the numerical schemes for the Eqs. (1.1) and established the error estimation of k-order of the DG-norm for velocity and L^2-norm for pressure. As highlighted in [12, 14], applying the standard finite element method to solve the such systems often leads to inaccurate and poor numerical solutions, that is, the accuracy of the approximate velocity depends on the pressure. Therefore, in order to surmount this problem or eliminate the influence of pressure on error estimates for the velocity, many scholars have done a lot of research and proposed some feasible and effective algorithms, such as the grad-div stabilization method [34, 45, 46], the pressure-robustness method [29, 37, 52], and the divergence-free method [12, 24, 58] to be utilized below.

On the other hand, most of the numerical methods for solving the Stokes equations with damping use triangular (simplicial) and quadrilateral meshes. However, with the fact that in regions of high curvature, the utilization of highly-stretched triangular or