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Analytic Regularity for a Singularly Perturbed Reaction-Convection-Diffusion Boundary Value Problem with Two Small Parameters

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Abstract. We consider a second order, two-point, singularly perturbed boundary value problem, of reaction-convection-diffusion type with two small parameters, and we obtain analytic regularity results for its solution, under the assumption of analytic input data. First, we establish classical differentiability bounds that are explicit in the order of differentiation and the singular perturbation parameters. Next, for small values of these parameters we show that the solution can be decomposed into a smooth part, boundary layers at the two endpoints, and a negligible remainder. Derivative estimates are obtained for each component of the solution, which again are explicit in the differentiation order and the singular perturbation parameters.

AMS subject classifications: 34E05, 34E10

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1 Introduction

Singularly perturbed problems and the numerical approximation of their solution have been studied extensively over the last few decades (see, e.g., the books

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[8, 9, 12] and the references therein). As is well known, a main difficulty in these problems is the presence of boundary layers in the solution, which appear due to fact that the limiting problem (i.e. when the singular perturbation parameter(s) tend to 0), is of different order than the original one, and the ('extra') boundary conditions can only be satisfied if the solution varies rapidly in the vicinity of the boundary – hence the name boundary layers.

In most numerical methods, high order derivatives of the solution appear in the error estimates, hence one should have a clear picture of how these derivatives grow with respect to the singular perturbation parameter(s). For low order numerical methods, such as finite differences (FD) or the h version of the finite element method (FEM), derivatives up to order 3 are usually sufficient. For high order methods such as the hp version of the FEM, derivatives of arbitrary order are needed, thus knowing how these behave with respect to the singular perturbation parameter(s) as well as the differentiation order, is necessary. Usually problems of convection-diffusion or reaction-diffusion type are studied separately and several researchers have proposed and analyzed numerical schemes for the robust approximation of their solution (see, e.g., [12] and the references therein). When there are two singular perturbation parameters present in the differential equation, the problem becomes reaction-convection-diffusion and the relationship between the parameters determines the 'regime' we are in (as shown in Table 1 ahead). In [3], the numerical solution to this problem was addressed, using the h version of the FEM as well as appropriate finite differences (see also [1, 2, 4, 11, 13, 16, 17]). Our interest in is high order hp FEM, hence we require information on all derivatives of the solution. In the present article we obtain information about the analytic regularity of the solution, using the method of asymptotic expansions (see also [6]), thus providing the tools for an hp FEM for the approximation of such problems.

The rest of the paper is organized as follows: in Section 2 we present the model problem and the regularity of its solution in terms of classical differentiability. Section 3 contains the asymptotic expansion for the solution, under the assumption that the singular perturbation parameters are small enough. We consider all possible relationships between the singular perturbation parameters, and establish derivative bounds which are explicit in the differentiation order as well as the singular perturbation parameters. We also comment on the transition between the regimes, in the final subsection of Section 3. Finally, in Section 4 we summarize our conclusions.

With $I \subset \mathbb{R}$ an open, bounded interval with boundary ∂I and measure $|I|$, we will denote by $C^k(I)$ the space of continuous functions on I with continuous derivatives up to order k. We will use the usual Sobolev spaces $W^{k,m}(I)$ of func-