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ASYMPTOTIC BEHAVIOR OF SOLUTION TO NONLINEAR DAMPED p-SYSTEM WITH BOUNDARY EFFECT

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Abstract. For the initial-boundary value problem to the 2×2 damped p-system with nonlinear source,

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta |u|^{q-1} u, \quad q \geq 2, \\ (v,u)|_{t=0} = (v_0,u_0)(x) \to (v_+,u_+) \ \text{ as } x \to +\infty, \\ u|_{x=0} = 0, \quad u_+ \neq 0, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

when $\beta > 0$, or $\beta < 0$ but $|\beta| < \frac{\alpha}{|u_+|^{q-1}}$, the solution $(v,u)(x,t)$ is proved to globally exist and converge to the solution of the corresponding porous media equations

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha \bar{u}, \\ \bar{v}|_{t=0} = \bar{v}_0(x) \to v_+ \ \text{ as } x \to +\infty, \\ \bar{u}|_{x=0} = 0, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

with a specially selected initial data $\bar{v}_0(x)$. The optimal convergence rates $\|\partial_x^k (v - \bar{v}, u - \bar{u})(t)\|_{L^2} = O(1)(t^{-\frac{2k+3}{4}}, t^{-\frac{2k+5}{4}})$, $k = 0, 1$, are also obtained, as the initial perturbation is in $L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$. If the initial perturbation is in the weighted space $L^{1,\gamma}(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$ with the best choice of $\gamma = \frac{1}{4}$, some new and much better decay rates are further obtained: $\|\partial_x^k (v - \bar{v})(t)\|_{L^2} = O(1)(1+t)^{-\frac{2k+3}{4} - \frac{\gamma}{2}}$, $k = 0, 1$. The proof is based on the technical weighted energy method combining with the Green function method. However, when $\beta < 0$ and $|\beta| > \frac{\alpha}{|u_+|^{q-1}}$, then the solution will blow up at a finite time. Finally, numerical simulations are carried out to confirm the theoretical results by using the central-upwind scheme. In particular, the interest phenomenon of coexistence of the global solution $v(x,t)$ and the blow-up solution $u(x,t)$ is observed and numerically demonstrated.

Key words. p-system of hyperbolic conservation laws, nonlinear damping, IBVP, porous equations, diffusion waves, asymptotic behavior, convergence rates, blow-up.

1. Introduction and Main Results

This is a series of study on the hyperbolic p-system with nonlinear source. In the first part [22], we investigated the asymptotic behavior of the solution for the Cauchy problem. Here, as the second part, we are going to treat the initial-boundary value problem. Namely, we study the 2×2 nonlinear damped p-system on the quadrant

$$(1) \qquad \begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = -\alpha u - \beta |u|^{q-1} u, \end{cases} \qquad (x,t) \in \mathbb{R}_+ \times \mathbb{R}_+,$$

with the initial-boundary conditions

$$(2) \qquad \begin{cases} (v,u)|_{t=0} = (v_0,u_0)(x) \to (v_+,u_+) \ \text{ as } x \to +\infty, \ x \in \mathbb{R}_+, \\ u|_{x=0} = 0. \end{cases}$$

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This model represents the compressible flow through porous media with nonlinear dissipative external force field in the Lagrangian coordinates. Here, $v = v(x,t) > 0$ is the specific volume, $u = u(x,t)$ is the velocity, the pressure $p(v)$ is a smooth function of v such that $p(v) > 0$, $p'(v) < 0$. As well-known in a hyperbolic system, the typical example in the case of a polytropic gas is $p(v) = v^{-\nu}$ with $\nu \geq 1$. The external term $-\alpha u - \beta|u|^{q-1}u$ appears in the momentum equation, where $\alpha > 0$ is a constant, $\beta \neq 0$ is another constant but can be either negative or positive. The term $-\alpha u$ is called the linear damping, and $-\beta|u|^{q-1}u$ with $q \geq 2$ is regarded as a nonlinear source to the linear damping $-\alpha u$. When $\beta > 0$, the term $-\beta|u|^{q-1}u$ is nonlinear damping, while, when $\beta < 0$, the term $-\beta|u|^{q-1}u$ is regarded as nonlinear accumulating. $v_+ > 0$ and u_+ are the state constants. For compatibility, we need $u_0(0) = 0$.

When $\beta = 0$, the system (1) is linear damping. The asymptotic behavior of the solution for the Cauchy problem or the IVBP for the linear damped 2×2 p-system has been extensively studied. In 1992, Hsiao and Liu [3, 4] first studied the Cauchy problem for the linearly damped p-system, and showed that the solution $(v,u)(x,t)$ converges to its diffusion wave $(\bar{v},\bar{u})(x/\sqrt{1+t})$, a self-similar solution to the following porous media equations

$$\begin{cases} \bar{v}_t - \bar{u}_x = 0, \\ p(\bar{v})_x = -\alpha\bar{u}, \end{cases} \quad \text{or} \quad \begin{cases} \bar{v}_t = -\frac{1}{\alpha}p(\bar{v})_{xx}, \\ p(\bar{v})_x = -\alpha\bar{u}, \end{cases} \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+,$$

in the form of $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1/2}, t^{-1/2})$. Since then, the convergence have been improved by Nishihara [24, 25] as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$ for the initial perturbation in $H^3(\mathbb{R})$, and then by Nishihara, Wang and Yang [28, 34] as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$ for the initial perturbation in $L^1(\mathbb{R}) \cap H^3(\mathbb{R})$. These convergence results need the initial perturbation around the specified diffusion wave and the wave strength both to be sufficiently small. Such restrictions were then partially released by Zhao [35], where the initial perturbation in L^∞-sense can be arbitrarily large but its first derivative must be sufficiently small, which implies that the wave must also be weak. Furthermore, when $v_+ = v_-$, Nishihara [26] improved the rates as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/2}\log t,\ t^{-2}\log t)$. Very recently, when $v_+ \neq v_-$, by a heuristic analysis, Mei [23] pointed out that the best asymptotic profile to the linearly damped p-system is the particular parabolic solution to the corresponding porous media equation with a specific initial data, rather than the self-similar solutions (the so-called nonlinear diffusion waves), and further proved the convergence as $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/2}\log t,\ t^{-2}\log t)$.

For the initial boundary problem on the quadrant in the case of linear damping (i.e., $\beta = 0$), the convergence to the diffusion waves with different boundary conditions has been studied respectively by Marcati and Mei [19] and by Nishihara and Yang [27] with the rate $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-3/4}, t^{-5/4})$ for the initial perturbation in $H^3(\mathbb{R}_+)$, respectively, and then, further improved to $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty} = O(1)(t^{-1}, t^{-3/2})$ by Marcati, Mei and Rubino [20] for the initial perturbation in $L^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$. Motivated by [35], the convergence has been improved by Jiang and Zhu [14] for the strong diffusion wave. Recently, Saind-Houari [31] claimed that the decay rate could be improved to $\|(v-\bar{v}, u-\bar{u})(t)\|_{L^\infty(\mathbb{R}_+)} = O(1)(t^{-1-\frac{\gamma}{2}}, t^{-\frac{3}{2}-\frac{\gamma}{2}})$, if the initial perturbation is in $L^{1,r}(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)$, where $L^{1,r}(\mathbb{R}_+)$ is a weighted L^1-space with the weight $(1+x)^\gamma$ and $0 \leq \gamma \leq 1$. However, this result is not correct in all cases, and the proof is also with some problems. In fact, the author just applied the well-known results from