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A Remark on Pál Type Interpolation on Non-Uniformly Distributed Nodes on the Unit Circle

Neha Mathur[1] and P. Mathur[2,*]

[1] *Department of Mathematics, Career Convent Degree College, Lucknow, India*
[2] *Department of Mathematics and Astronomy, Lucknow University, Lucknow, India*

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Abstract. In this paper we study the problem of explicit representation and convergence of Pál type $(0;1)$ interpolation and its converse, with some additional conditions, on the non-uniformly distributed nodes on the unit circle obtained by projecting the interlaced zeros of $P_n(x)$ and $P_n'(x)$ on the unit circle. The motivation to this problem can be traced to the recent studies on the regularity of Birkhoff interpolation and Pál type interpolations on non-uniformly distributed zeros on the unit circle.

Key Words: Pál type interpolation, non-uniformly distributed set of points on unit circle, Legendre polynomials.

AMS Subject Classifications: 41A05, 41A10, 41A25

1 introduction

Let $\{x_{k,n}\}_{k=1}^{n}$ and $\{y_{k,n-1}\}_{k=1}^{n-1}$ be the zeros of $P_n(x)$ and $P_n'(x)$ respectively in $[-1,1]$, where $P_n(x)$ is the nth Legendre polynomial, which are interlaced such that

$$-1=x_{0,n}<x_{1,n}<y_{1,n-1}<\cdots<x_{n-1,n}<y_{n-1,n-1}<x_{n,n}<x_{n+1,n}=1. \tag{1.1}$$

We project these points on the unit circle by the inverse of the transformation

$$x=\frac{1}{2}(z+z^{-1}).$$

Let $\{z_{k,2n+2}\}_{k=1}^{2n}$ and $\{w_{k,2n-2}\}_{k=1}^{2n-2}$ be the transformations of $\{x_{k,n}\}_{k=1}^{n}$ and $\{y_{k,n-1}\}_{k=1}^{n-1}$ respectively on the unit circle and $z_{0,2n+2}=-1$, $z_{2n+1,2n+2}=1$. The set of points

$$\Lambda=\{z_{k,2n+2}\}_{k=0}^{2n+1}\cup\{w_{k,2n-2}\}_{k=1}^{2n-2}, \tag{1.2}$$

*Corresponding author. *Email addresses:* `neha_mathur13@yahoo.com` (N. Mathur), `pankaj_mathur14@yahoo.co.in` (P. Mathur)

thus obtained on the unit circle is non-uniformly distributed.

A Pál type $(0;1)$ interpolation on Λ means the determination of a polynomial R_n of minimum possible degree when the function values are prescribed on $\{z_{k,2n+2}\}_{k=0}^{2n+1}$ and the first derivatives are prescribed on $\{w_{k,2n-2}\}_{k=1}^{2n-2}$, i.e.,

$$R_n(z_{k,2n+2})=\alpha_{k,2n+2}, \qquad k=0,1,2\cdots,2n,2n+1, \tag{1.3a}$$
$$R'_n(w_{k,2n-2})=\beta_{k,2n-2}, \qquad k=1,2\cdots,2n-2, \tag{1.3b}$$

where $\{\alpha_{k,2n+2}\}_{k=0}^{2n+1}$ and $\{\beta_{k,2n-2}\}_{k=1}^{2n-2}$ are arbitrary given complex numbers. In this paper, it has been shown that Pál type $(0;1)$ interpolation on Λ is regular i.e., there exists a unique polynomial of degree $\leq 4n-1$ satisfying the conditions (1.3a) and (1.3b). The explicit representation of the interpolatory polynomial has been obtained and a convergence theorem has also been proved for the same. If $\{z_{k,2n+2}\}_{k=0}^{2n+1}$ and $\{w_{k,2n-2}\}_{k=1}^{2n-2}$ are interchanged in (1.3a) and (1.3b) respectively (the converse of the above problem) then it has been shown that the Pál type $(0;1)$ interpolation is regular together with some additional interpolatory conditions. The explicit representation of the corresponding interpolatory polynomial and its convergence has also been dealt with. Further it has been shown, by numerical calculations, that the maximum absolute error in (i) the Lagrange interpolation for the simple function $f(z)=\exp(z)$, $z\in\mathbb{C}$ and (ii) the barycentric [2] form of Lagrange interpolation for the simple function $f(z)=1/(1+z^2)$, $z\in\mathbb{C}$ is least when the interpolation points are chosen to be the points of the set Λ given by (1.2) in comparison to that obtained when the points of interpolation are (i) projected zeros of the Chebyshev polynomial $T_N(x)$, rescaled so that the first and last zeros coincide, respectively, with -1 and 1, to the unit circle and (ii) the N^{th} roots of unity.

The study of interpolation processes on the unit circle was initiated in 1960 by O. Kiš [12], when he considered the $(0,2)$ and $(0,1,\cdots,r-2,r)$ interpolations, for any integer $r\geq 2$ on nth roots of unity. Since then several mathematicians have taken up the study of various interpolatory problems on the unit circle. Considerable literature has come up on the subject of lacunary and Pál type interpolations on the roots of unity [13]. Recently, the regularity of Birkhoff interpolation and Pál type interpolation on non-uniformly distributed nodes on the unit circle has been attracting much attention, which motivated us to study the above problem.

R. Brück [4] has studied the convergence of Lagrange interpolation of a function on the nodes $z_{kn}^{\alpha}=T_\alpha(w_{kn})$, $k=1(1)2n$, where T_α is a Möbius transform of a unit disk into itself and $w_{kn}=\exp(2\pi ik/(2n+1))$, $n\geq 0$. In [3] Bokhari et al. and in [10] de Bruin et al. have obtained the regularity of certain interpolation problems on the zeros of $(z-\xi)Q(z)$, where $Q(z)$ is a polynomial, whose all zeros lie on the unit circle. They have also determined the range of the values of ξ in the complex plane. In [11], Dikshit has shown the regularity of certain Pál type interpolation problems involving Möbius transforms of the zeros of z^n+1 and z^n-1 along with an additional point ξ or two additional points ξ and 1. For more references of recent works in this direction, we refer to [3–11,14].

In another paper, S. Xie [17] considered among others regularity of $(0,1,\cdots,r-2,r)$ and